

COMMENSURABILITY OF GROUPS QUASI-ISOMETRIC TO RAAG'S

JINGYIN HUANG

ABSTRACT. Let G be a right-angled Artin group with defining graph Γ and let H be a finitely generated group quasi-isometric to $G(\Gamma)$. We show if G satisfies (1) its outer automorphism group is finite; (2) Γ does not contain any induced 4-cycle; (3) Γ is star-rigid; then H is commensurable to G . We show condition (2) is sharp in the sense that if Γ contains an induced 4-cycle, then there exists an H quasi-isometric to G but not commensurable to G . Moreover, one can drop condition (1) if H is a uniform lattice acting on the universal cover of the Salvetti complex of G . As a consequence, we obtain a conjugation theorem for such uniform lattices. The ingredients of the proof include a blow-up building construction in [44] and a Haglund-Wise style combination theorem for certain class of special cube complexes. However, in most of our cases, relative hyperbolicity is absent, so we need new ingredients for the combination theorem.

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1. INTRODUCTION

1.1. Background and overview. It is well-known that if a finitely generated group is quasi-isometric to a free group, then it is commensurable to a free group. In this paper, we seek higher dimensional version of this fact in the class of right-angled Artin groups. Recall that given a finite simplicial graph Γ , the *right-angled Artin group* (RAAG) $G(\Gamma)$ with defining graph Γ is given by the following presentation:

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$$\{v_i \in \text{Vertex}(\Gamma) \mid [v_i, v_j] = 1 \text{ if } v_i \text{ and } v_j \text{ are joined by an edge}\}.$$

This class of groups has deep connections with a lot of other objects. Wise has proposed a program for embedding certain groups virtually into RAAG's ([63, Section 6]). One highlight in this direction is the recent solution of virtual Haken conjecture, which relies on the embedding of fundamental groups of 3-dimensional hyperbolic manifolds into RAAG's [2, 59]. Other examples include Coxeter groups [38], certain random groups [52] and small cancellation groups [62], certain classes of 3-manifold groups [53, 54, 51, 32], hyperbolic free-by-cyclic groups [33, 34], one relator groups with torsion and limit groups [59] etc.

A theory of special cube complexes has been developed along the way [61, 36, 37, 39], which can be viewed as a higher-dimensional analogue of earlier work of Stallings [57] and Scott [55]. In particular, it offers a geometric way to study finite index subgroups of virtually special groups.

This paper is motivated by the following question, which falls into Gromov's program of classifying groups and metric spaces up to quasi-isometry.

Question 1.1. *Let H be a finitely generated group quasi-isometric to $G(\Gamma)$. When is H commensurable to $G(\Gamma)$? Which $G(\Gamma)$ is rigid in the sense that any such H quasi-isometric to $G(\Gamma)$ is commensurable to $G(\Gamma)$?*

This question naturally leads to studying finite index subgroups of $G(\Gamma)$. The aforementioned theory of special cube complexes turns out to be one of our main ingredients. We will give a class of quasi-isometrically rigid RAAG's, based upon a series of previous works [10, 9, 42, 41, 44].

Before going into detailed discussion about RAAG's, we would like to compare RAAG's with other objects studied under Gromov's program. Previously quasi-isometry classification or rigidity results of spaces and groups can be very roughly divided into the case where various notions of non-positive curvature conditions are satisfied (like relative hyperbolicity, $CAT(0)$, coarse median etc), and the case where the non-positive curvature condition is absent (like certain solvable groups and graph of groups etc). The non-positively curved world can be further divided into the hyperbolic case (including relative hyperbolic with respect to flats), and the higher rank case, where there are lot of higher dimensional flats in the space, and they have non-trivial intersection pattern. Most RAAG's belong to the last case. Other higher rank cases include (1) higher rank lattices in semisimple Lie groups [50, 27, 26]; (2) mapping class groups and related spaces [28, 14, 13, 12, 6, 40]; (3) certain 3-manifold groups and their generalizations [46, 8, 7, 29]. The rigidity of these spaces often depends on the complexity of the intersection pattern of flats inside the space. The classes of RAAG's studied in this paper will be more "rigid" than (3), but more "flexible" than (1) and (2), see [9, 41, 44] for a detailed discussion.

In general, quasi-isometries of $G(\Gamma)$ may not be of bounded distance from isometries, even if they are equivariant with respect to a cobounded group action. However, it is shown in [9, 41] that when the outer automorphism group $\text{Out}(G(\Gamma))$ is finite, every quasi-isometry is of bounded distance from a canonical representative which preserves standard flats of $G(\Gamma)$. Based on this, we show in [44] that if H is quasi-isometric to $G(\Gamma)$ with $\text{Out}(G(\Gamma))$ finite, then H acts properly and cocompactly on a $CAT(0)$ cube complex X which is closely related to the universal cover of the Salvetti complex (Section 2.1) of $G(\Gamma)$.

Thus the problem is related to the commensurability classification of uniform lattices in $\text{Aut}(X)$ for a $CAT(0)$ cube complex X . This is well-understood when

X is a tree [4]. In the case of right-angled buildings, Haglund observed a relation between commensurability of lattices and separability of certain subgroups [35], this together with [2] give commensurability results for some classes of uniform lattices in hyperbolic right-angled buildings. However, when X is not hyperbolic, there exist uniform lattices with very exotic behaviour [60, 17]. The cases we are interested in is not (relatively) hyperbolic, and the above literature will serve as a guideline for putting suitable conditions on Γ .

We close this section with a summary of previously known examples and non-examples of classes of RAAG's which satisfy commensurability rigidity in the sense of Question 1.1. The rigid ones include

- The free group F_m of m -generators [56, 25, 47], [31, 1.C].
- The free Abelian groups \mathbb{Z}^n [30, 3].
- $F_m \times \mathbb{Z}^n$ [58].
- Free products of free groups and free Abelian groups [5].

The non-rigid classes of RAAG's include

- $F_m \times F_\ell$ with $m, \ell \geq 2$ [60, 17].
- $G(\Gamma)$ with Γ being a tree of diameter ≥ 3 [8].

1.2. Main results. All graphs in this section are simple. A finite graph Γ is *star-rigid* if for any automorphism α of Γ , α is identity whenever it fixes a closed star of some vertex $v \in \Gamma$ point-wise. Two groups H and G are *commensurable* if there exist finite index subgroups $H' \leq H$, $G' \leq G$ such that H' and G' are isomorphic.

Theorem 1.2. *Suppose Γ is a graph such that*

- (1) Γ is star-rigid;
- (2) Γ does not contain induced 4-cycle;
- (3) $\text{Out}(G(\Gamma))$ is finite.

Then any finite generated group quasi-isometric to $G(\Gamma)$ is commensurable to $G(\Gamma)$.

One may find it helpful to think about the case when Γ is a pentagon. Condition (2) is sharp in the following sense.

Theorem 1.3. *Suppose Γ is a graph which contains an induced 4-cycle. Then there exists a finitely generated group H which is quasi-isometric to $G(\Gamma)$ but not commensurable to $G(\Gamma)$.*

It follows that there are plenty of RAAG's with finite outer automorphism group which are not rigid in the sense of Question 1.1. This is quite different from the conclusion in the internal quasi-isometry classification of RAAG's [9, 41].

Remark 1.4. It is natural to ask how much portion of RAAG's which are rigid in the sense of Question 1.1 are characterized by Theorem 1.2. This is related to a graph theoretic question as follows. There is a 1-1 correspondence between finite simplicial graphs up to graph isomorphisms and RAAG's up to group isomorphisms [24]. Thus it makes sense to talk about random RAAG's by considering random graphs. It was shown in [21, 23] that a random RAAG satisfies (3). Moreover, since a finite random graph is asymmetric, it satisfies (1). However, we ask whether (1) and (3) are also generic properties among finite simplicial graphs which do not have induced 4-cycles. A positive answer to this question, together with Theorem 1.2 and Theorem 1.3, would mean almost all RAAG's which are rigid in the sense of Question 1.1 are characterized by Theorem 1.2.

Condition (3) of Theorem 1.2 is motivated by the study of asymptotic geometry of RAAG's. It turns out that $G(\Gamma)$ satisfies a form of vertex rigidity ([9, Theorem 1.6] and [41, Theorem 4.15]) if and only if $\text{Out}(G(\Gamma))$ is finite. The motivation for condition (1) and condition (2) is explained after Corollary 1.6.

Let $S(\Gamma)$ be the Salvetti complex of $G(\Gamma)$ (Section 2.1) and let $X(\Gamma)$ be the universal cover of $S(\Gamma)$. The following is a simpler version of Theorem 1.2 in the case of lattices in $\text{Aut}(X(\Gamma))$.

Theorem 1.5. *Suppose Γ is a graph such that*

- (1) Γ is star-rigid;
- (2) Γ does not contain induced 4-cycle.

Let H be a group acting geometrically on $X(\Gamma)$ by automorphisms. Then H and $G(\Gamma)$ are commensurable. Moreover, let $H_1, H_2 \leq \text{Aut}(X(\Gamma))$ be two uniform lattices. Then there exists $g \in \text{Aut}(X(\Gamma))$ such that $gH_1g^{-1} \cap H_2$ is of finite index in both gH_1g^{-1} and H_2 .

Let $H \leq \text{Aut}(X(\Gamma))$. Recall that the *commensurator* of H in $\text{Aut}(X(\Gamma))$ is $\{g \in \text{Aut}(X(\Gamma)) \mid gHg^{-1} \cap H \text{ is of finite index in } H \text{ and } gHg^{-1}\}$. Then Theorem 1.5 and [36, Theorem I] imply the following arithmeticity result.

Corollary 1.6. *Let Γ be as in Theorem 1.5. Then the commensurator of any uniform lattice in $\text{Aut}(X(\Gamma))$ is dense in $\text{Aut}(X(\Gamma))$.*

Recall that edges in $X(\Gamma)$ can be labelled by vertices of Γ (see Section 2.1). Condition (1) is to guarantee that every H has label-preserving finite index subgroup. We caution the reader that this condition is not equivalent to the “type preserving” condition introduced in [35] in the studying of right-angled buildings, though these two conditions have similar flavour.

Theorem 1.5 is a consequence of the following result.

Theorem 1.7. *Suppose Γ is a graph which does not contain induced 4-cycle. Let H be a group acting geometrically on $X(\Gamma)$ by label-preserving automorphisms. Then H is commensurable to $G(\Gamma)$.*

For a more general version of Theorem 1.7, see Theorem 7.11.

The no induced 4-cycle condition is motivated by the examples of exotic groups acting geometrically on the product of two trees [60, 17]. It is not hard to see that $X(\Gamma)$ contains an isometrically embedded product of two trees of infinitely many ends if and only if Γ contains an induced 4-cycle. It turns out that one can modify their examples such that the exotic action on the product of two trees extends to an exotic action on $X(\Gamma)$, which gives a converse to Theorem 1.7.

Theorem 1.8. *Let Γ be a graph which contains an induced 4-cycle. Then there exists a torsion free group H acting geometrically on $X(\Gamma)$ by label-preserving automorphisms such that H is not residually finite. In particular, H and $G(\Gamma)$ are not commensurable.*

Note that even in the case where Γ is a 4-cycle, the above theorem is not completely trivial, since the examples mentioned above in [60, 17] are not label-preserving. Also Theorem 1.3 follows from Theorem 1.8.

1.3. Sketch of proof and organization of the paper. We refer to Section 2.1 for our notations. Let $G(\Gamma)$ be a RAAG with finite outer automorphism group and let H be a group quasi-isometric to $G(\Gamma)$. For simplicity we assume H is torsion free and H acts geometrically on $X(\Gamma)$. In general, H admits a geometric model which is quite similar to $X(\Gamma)$, which is discussed in Section 3 and Section 4.

Step 1: We orient edges in $S(\Gamma)$ and label them by vertices of Γ . This lifts to $G(\Gamma)$ -equivariant orientation and labelling of edges in $X(\Gamma)$. We are done if H happens to preserve both the orientation and labelling. By a simple observation (Lemma 4.6), we can assume H acts on $G(\Gamma)$ in a label-preserving way by passing to a finite index subgroup. However, the issue with orientation is more serious.

We induct on the complexity of the underlying graph, and assume Theorem 1.7 holds for any proper subgraph of Γ . Pick a vertex $v \in \Gamma$ and let $\Gamma' \subset \Gamma$ be the induced subgraph spanned by vertices in $\Gamma \setminus \{v\}$. Then there is a canonical embedding $S(\Gamma') \rightarrow S(\Gamma)$. Note that $G(\Gamma)$ is an HNN-extension of $G(\Gamma')$ along the subgroup $G(lk(v)) \subset G(\Gamma')$. Let T be the associated Bass-Serre tree. Alternatively, T is obtained from $X(\Gamma)$ by collapsing edges which are in the lifts of $S(\Gamma')$.

Since H is label-preserving, there is an induced action $H \curvearrowright T$. Up to passing to a subgroup of index 2, we assume H acts on T without inversion. This induces a graph of groups decomposition of H and a graph of spaces decomposition of $K = X(\Gamma)/H$. Each vertex group acts geometrically on $X(\Gamma')$ by label-preserving automorphisms (since the universal cover of each vertex space is isometric to $X(\Gamma')$), hence it is commensurable to $G(\Gamma')$ by the induction assumption. It follows that each vertex space has a finite sheet cover which is a special cube complex. At this point, the reader is not required to know what is exactly a special cube complex. One may perceive it as a cube complex with some nice combinatorial features and we will explain the relevant properties later.

Each cover of K has an induced graph of spaces structure. We claim there exists a finite sheet cover $\tilde{K} \rightarrow K$ such that each vertex space of \tilde{K} is a special cube complex. It is not hard to deduce the above theorem from this claim. The relation between specialness and commensurability is discussed in Section 4.3.

Any edge group of T/H acts geometrically on $X(lk(v))$. In general they could be very complicated, however, their intersections are controlled as follows.

Lemma 1.9. *Suppose Γ has no induced 4-cycle. Then the largest intersection of edge groups and their conjugates in a vertex group is virtually a free Abelian group.*

Step 2: For simplicity, we assume K only has two vertex spaces L and R , and one edge space $E \subset K$ such that the two ends of $E \times [0, 1]$ are attached isometrically to locally convex subcomplexes $E_L \subset L$ and $E_R \subset R$ respectively. Let $L_1 = L \cup (E \times [0, 1/2])$ and $R_1 = R \cup (E \times [1/2, 1])$.

It follows from the work of Haglund and Wise [39] that if $\pi_1(K)$ is hyperbolic, and if $\pi_1(E_L)$ and $\pi_1(E_R)$ are malnormal in $\pi_1(L)$ and $\pi_1(R)$ respectively, then K has the desired finite sheet cover. Later the malnormality is dropped in [59] by using the cubical small cancellation theory, and hyperbolicity is relaxed to relative hyperbolicity with respect to Abelian subgroups. However, in general it is easy to give counterexamples without any hyperbolicity assumption. Moreover, these works did not take quasi-isometry invariance into consideration.

In most of our cases, both relative hyperbolicity and malnormality fail. However, K has more structure than a generic special cube complex and malnormality does not fail in a terrible way (Lemma 1.9). Due to lack of hyperbolicity, we will get

around the cubical small cancellation or Dehn filling argument, and use a different collapsing argument based on the blow-up building construction in [44], which also applies to group quasi-isometric to $G(\Gamma)$.

Let $L' \rightarrow L$ be a finite sheet special cover. It also induces a finite sheet cover $L'_1 \rightarrow L_1$. Similarly we define R' and R'_1 . Usually there are more than one lifts of E_L in L' , each lift gives rise to a half-tube in L'_1 . It suffices to match half-tubes in L'_1 with half-tubes in R'_1 . Suppose there are bad half-tubes in L'_1 which do not match up with half-tubes in R'_1 . It is natural to ask whether it is possible to pass to finite sheet covers of L'_1 such that bad half-tubes are modified in the cover while good half-tubes remain unchanged.

One ideal situation is the following. Suppose $A_1, A_2 \subset L'$ are elevations of E_L and suppose there exists a retraction $r : L' \rightarrow A_1$ such that $r_*(\pi_1(A_2))$ is trivial. Then there exists cover $L'' \rightarrow L'$ which realizes any further cover of A_1 without changing A_2 . We want to achieve at least some weaker version of this ideal situation. Since L' is a special cube complex, L' has a finite sheet cover which retracts onto A_1 . This is constructed in [37], in which it is called the *canonical retraction*. Then we need to look at the images of the lifts of A_2 under this retraction. Due to the failure of malnormality, $r_*(\pi_1(A_2))$ is in generally non-trivial, however, Lemma 1.9 suggested that it is reasonable to control the retraction image such that it is not more complicated than a torus.

The rough idea is that we first collapse all the tori in K . Then the tubes become collapsed tubes and the previous statement is equivalent to that the projection of a collapsed tube to another collapsed tube is contractible. The reader may notice that if we collapse all the tori in K , then the resulting space is a point. In order to make this idea work, we first exploded K with respect to the intersection pattern of the tori in K , then collapse the tori. The resulting space K' is a space which encodes the intersection pattern of tori in K . This is done in the setting of blow-up building developed in [44]. Moreover, this also works in the case when H is a group quasi-isometric to $X(\Gamma)$.

By killing certain holonomy in K (see Section 5), K' becomes a special cube complex. Moreover, one deduce that $\pi_1(K')$ is Gromov-hyperbolic from the no induced 4-cycle condition (see Section 6.1). Now we are in a situation to apply the work of Haglund and Wise on hyperbolic special cube complex.

Step 3: While matching the tubes, we also need to keep track of finer information about these retraction tori, such as the length of the circles in the tori, and how other circles retracts onto a particular circle. It turns out that the construction in [37] does not quite preserve this information since the canonical completion is too large in some sense (Remark 2.17). We need a modified version of completion and retraction, which is discussed in Section 6.2.

Step 4: Given that the retraction images are tori, and the retraction preserves finer combinatorial information of tori in K , we construct the desired cover of K in Section 7.2. The argument is a modified version of [39, Section 6].

Step 5: We show how to drop the finite outer automorphism condition in the case of group acting geometric on $X(\Gamma)$. We will explode K in a different way and decompose it into suitable vertex spaces and edge spaces. See Section 8.

Step 6: When there is an induced 4-cycle in Γ , the largest intersection of edges groups and their conjugates in a vertex group may contain a free group of rank 2.

In this case it is impossible for K to be virtually special in general. The counterexamples are given in Section 9.

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2. PRELIMINARIES

2.1. Right-angled Artin groups and Salvetti complexes. We refer to [20] for background on right-angled Artin groups. Throughout this section, Γ will be a finite simplicial graph.

Definition 2.1 (Salvetti complex). Denote the vertex set of Γ by $\{v_i\}_{i=1}^m$. We associated each v_i with a standard circle $\mathbb{S}_{v_i}^1$ and choose a base point $p_i \in \mathbb{S}_{v_i}^1$. Let $\mathbb{T}^m = \prod_{i=1}^m \mathbb{S}_{v_i}^1$. Then \mathbb{T}^m has a natural cube complex structure. Then Δ gives rise to a subcomplex $T_\Delta = \prod_{v_i \in v(\Delta)} \mathbb{S}_{v_i}^1 \times \prod_{v_i \notin v(\Delta)} \{p_i\}$. Then $S(\Gamma)$ is defined to be the subcomplex of \mathbb{T}^m which is the union of all T_Δ 's with Δ ranging over all clique subgraphs of Γ .

$S(\Gamma)$ is a non-positively curved cube complex whose 2-skeleton is the presentation complex of $G(\Gamma)$, so $\pi_1(S(\Gamma)) \cong G(\Gamma)$. The closure of each k -cell in $S(\Gamma)$ is a k -torus, which is called a *standard torus*. A standard torus of dimension 1 is also called a *standard circle*. The *dimension* of $G(\Gamma)$ is the dimension of $S(\Gamma)$. Recall that $\Gamma' \subset \Gamma$ is an *induced subgraph* if vertices of Γ' are adjacent in Γ' if and only if they are adjacent in Γ . Each induced subgraph $\Gamma' \subset \Gamma$ gives rise to an isometric embedding $S(\Gamma') \rightarrow S(\Gamma)$. The universal cover of $S(\Gamma)$ is a $CAT(0)$ cube complex, which we denote by $X(\Gamma)$. We label standard circles of $S(\Gamma)$ by vertices of Γ , and this lifts to a $G(\Gamma)$ -invariant edge labelling of $X(\Gamma)$.

Definition 2.2. (Γ' -components) Let K be a cube complex. Suppose edges of K are labelled by vertices of Γ . Pick a induced subgraph $\Gamma' \subset \Gamma$. A Γ' -*component* $L \subset K$ is a subcomplex such that

- (1) L is connected.
- (2) Each edge in L is labelled by a vertex in Γ' . Moreover, for each vertex $v \in \Gamma'$, there exists an edge in L which is labelled by v .
- (3) L is maximal with respect to (1) and (2).

Here Γ' is called the *support* of L .

Note that an \emptyset -component is a vertex in K . If Γ' is a complete graph of k vertices, Γ' -components of $X(\Gamma)$ are isometrically embedded k dimensional Euclidean spaces. They are called *standard flats*. When $k = 1$, we also call them *standard geodesics*. Vertices in $X(\Gamma)$ are understood to be 0-dimensional standard flats. Also note that in order to define Γ' -components in $S(\Gamma)$ and $X(\Gamma)$, the second part of Definition 2.2 (2) is not necessary, since for each vertex x in $S(\Gamma)$ or $X(\Gamma)$, and for each vertex $v \in \Gamma$, there exists a v -labelled edge containing x . However, we will encounter other complexes later where this property is not true.

Lemma 2.3. *Suppose K is non-positively curved and parallel edges of K are labelled by the same vertex. Then each Γ' -component L is locally convex. In particular, if K is $CAT(0)$, then L is $CAT(0)$.*

Proof. It suffices to check for each vertex $x \in L$, if a collection of edges $\{e_i\}_{i=1}^n$ emanating from x span a cube in the ambient complex, then this cube is in L . However, this follows from the fact that parallel edges have the same label, and the maximality of L . \square

The following object was first introduced in [48].

Definition 2.4 (extension complex). The *extension complex* $\mathcal{P}(\Gamma)$ of a finite simplicial graph Γ is defined as follows. The vertices of $\mathcal{P}(\Gamma)$ are in 1-1 correspondence with the parallel classes of standard geodesics in $X(\Gamma)$. Two distinct vertices $v_1, v_2 \in \mathcal{P}(\Gamma)$ are connected by an edge if and only if there is a standard geodesic l_i in the parallel class associated with v_i ($i = 1, 2$) such that l_1 and l_2 span a standard 2-flat. Then $\mathcal{P}(\Gamma)$ is defined to be the flag complex of its 1-skeleton, namely we build $\mathcal{P}(\Gamma)$ inductively from its 1-skeleton by filling a k -simplex whenever we see the $(k - 1)$ -skeleton of a k -simplex.

Since each complete subgraph in the 1-skeleton of $\mathcal{P}(\Gamma)$ gives rise to a collection of mutually orthogonal standard geodesics lines in $X(\Gamma)$, there is a 1-1 correspondence between k -simplexes in $\mathcal{P}(\Gamma)$ and parallel classes of standard $(k + 1)$ -flats in $X(\Gamma)$. For standard flat $F \subset X(\Gamma)$, we denote the simplex in $\mathcal{P}(\Gamma)$ which corresponds to standard flats parallel to F by $\Delta(F)$.

Each vertex $v \in \mathcal{P}(\Gamma)$ is labelled a vertex of Γ in the following way. Pick standard geodesic $\ell \subset X(\Gamma)$ such that $\Delta(\ell) = v$ and pick edge $e \subset \ell$. We label v by the label of e . Note that this does not depend on the choice of ℓ in the parallel class, and the edge inside ℓ . This labelling induces a map from the 0-skeleton of $\mathcal{P}(\Gamma)$ to vertices in Γ , which can be extended to a simplicial map $\mathcal{P}(\Gamma) \rightarrow F(\Gamma)$, here $F(\Gamma)$ is the flag complex of Γ .

We refer to [50, Section 2.3.3] for the definition and properties of the *parallel set* of a convex subset in a $CAT(0)$ space.

Definition 2.5. For vertex $v \in \mathcal{P}(\Gamma)$, the *v-parallel set*, which we denote by P_v , is the parallel set of a standard geodesic $\ell \subset X(\Gamma)$ such that $\Delta(\ell) = v$. Note that P_v does not depend on the choice of the standard geodesic ℓ in the parallel class.

Definition 2.6. Pick induced subgraph $\Gamma' \subset \Gamma$. The *link* of Γ' , denoted by $lk(\Gamma')$, is the induced subgraph spanned by vertices which are adjacent to every vertex in Γ' . The *closed star* of Γ' , denoted by $St(\Gamma')$, is the induced subgraph spanned by vertices in Γ' and $lk(\Gamma')$.

Let K be a polyhedral complex and pick $x \in K$. The *geometrical link* of x in K , denoted by $Lk(x, K)$, is the object defined in [15, Chapter I.7.15].

In general, these two notion of links do not agree on graphs.

Lemma 2.7. ([41, Lemma 3.4]) *Let K be a Γ' -component in $X(\Gamma)$. Then the parallel set P_K of K is exactly the $St(\Gamma')$ -component containing K . Moreover, P_K admits a splitting $P_K = K \times K^\perp$, where K^\perp is isomorphic to a $lk(\Gamma')$ -component.*

In particular, for vertex $v \in \mathcal{P}(\Gamma)$, P_v is the $St(\bar{v})$ -component that contains ℓ , where ℓ is a standard geodesic with $\Delta(\ell) = v$ and $\bar{v} \in \Gamma$ is the label of edges in ℓ .

2.2. Special cube complex. We refer to Section 2.A and Section 2.B of [39] for background about cube complexes and hyperplanes. An edge is *dual* to a hyperplane if it has nonempty intersection with the hyperplane. A hyperplane h is

2-sided if it has a small neighbourhood which is a trivial bundle over h . Two edges are *parallel* if they are dual to the same hyperplane.

Let X be a cube complex and pick a two-sided hyperplane $h \subset X$. Then the parallelism induces a well-defined orientation for edges dual to h . For each vertex $v \in X$, let g_v be the graph made of all dual edges of h that contains v (g_v could be empty). Then the geometric link $Lk(v, g_v)$ is a discrete graph and each of its vertices is either incoming or outgoing depending on the orientations of edges.

We say h *self-osculates* if

- (1) h is two-sided and embedded;
- (2) there exists vertex $v \in X$ such that $Lk(v, g_v)$ has more than one points.

In this case, h *directly self-osculates* if there exist vertex $v \in X$ such that $Lk(v, g_v)$ has at least two vertices which are both incoming or both outgoing, otherwise, h *indirectly self-osculates*. For example, a circle with one edge has an indirectly self-osculating hyperplane.

Let h_1 and h_2 be a pair of embedded two-sided hyperplanes. Then they *interosculate* if there exist edges a_i, b_i dual to h_i ($i = 1, 2$), and vertices $v_a \in a_1 \cap a_2$, $v_b \in b_1 \cap b_2$ such that

- (1) there exist a vertex in $Lk(v_a, a_1)$ and a vertex in $Lk(v_a, a_2)$ which are adjacent in $Lk(v_a, X)$ (note that if a_1 is not embedded, then $Lk(v_a, a_1)$ has two points);
- (2) there exist a vertex in $Lk(v_b, b_1)$ and a vertex in $Lk(v_b, b_2)$ which are not adjacent in $Lk(v_b, X)$.

Definition 2.8. A non-positively curved cube complex X is *special* if

- (1) Each hyperplane is 2-sided and embedded.
- (2) No hyperplane directly self-osculate.
- (3) No two hyperplanes interosculate.

X is *directly special* if X is special and no hyperplane of X self-osculate.

Theorem 2.9. [39, Lemma 2.6] *Let X be a non-positively curved cube complex. Then X is special if and only if there exists a local-isometry $X \rightarrow S(\Gamma)$ for some Salvetti complex $S(\Gamma)$.*

Lemma 2.10. [39] *Let X be a compact special cube complex. Then X has a finite cover which is directly special. Moreover, being directly special is preserved under passing to any further cover.*

2.3. Canonical completions and retractions. We follow [39, Section 3].

Given compact special cube complexes A and X , and a locally isometry $A \rightarrow X$, one can construct a finite cover of X that contains a copy of A . This finite cover is called the *canonical completion* of the pair (A, X) and is denoted by $C(A, X)$. Moreover, there is a *canonical retraction* map $r : C(A, X) \rightarrow A$. Now we describe the construction of $C(A, X)$ and the retraction map r .

Case 1: X is a circle with a single edge. Then each connected component of A is either a point, or a circle, or a path. We attach an extra edge to each component which is not a circle to make it a circle. The resulting space is denoted by $C(A, X)$. It is clear that $C(A, X)$ is finite cover of X and $A \subset C(A, X)$. Moreover, there is a retraction $C(A, X) \rightarrow A$ by sending each extra edge to the component of A it was attached along.

Case 2: X is a wedge of finitely many circles $\{c_i\}_{i=1}^n$. Let A_i be the inverse image of c_i under $A \rightarrow X$. We define $C(A, X)$ to be the union of all $C(A_i, c_i)$'s identified along their vertex sets, and define the canonical retraction $r : C(A, X) \rightarrow A$ to be the map induced by $C(A_i, c_i) \rightarrow A_i$. It is still true that $C(A, X) \rightarrow X$ is a finite sheet covering map and $A \subset C(A, X)$.

Case 3: X is the Salvetti complex of some RAAG. Then the 1-skeleton X^1 of X is a wedge of circles. It turns out that there is a natural way to attach higher dimensional cells to $C(A^1, X^1)$ to obtain a non-positively curved cube complex $C(A, X)$ (see [11, Theorem 2.6]). Moreover, the covering map $C(A^1, X^1) \rightarrow X^1$ and the retraction $C(A^1, X^1) \rightarrow A^1$ extent to a covering map $C(A, X) \rightarrow X$ and a retraction $C(A, X) \rightarrow A$. The following is immediate from our construction.

Lemma 2.11. *For $i = 1, 2$, let $B_i \rightarrow X_i$ be a local isometry from B_i to a Salvetti complex. Then the canonical completion $C(B_1 \times B_2, X_1 \times X_2)$ with respect to the product of these two local isometries is naturally isomorphic (as cube complexes) to $C(B_1, X_1) \times C(B_2, X_2)$.*

Case 4: X is any compact special cube complex. Let Γ be a graph such that its vertex set corresponds to the hyperplanes in X , and two vertex are adjacent if the corresponding hyperplanes cross each other. Such graph is called the *intersection graph* of X . Suppose $R = S(\Gamma)$. Since X is special, there is a local isometry $X \rightarrow R$, which induces a local isometry $A \rightarrow R$. We define $C(A, X)$ to be the pull-back of the covering map $C(A, R) \rightarrow R$ which fits into the following commuting diagram:

$$\begin{array}{ccccc} C(A, X) & \longrightarrow & C(A, R) \\ & \downarrow & \downarrow \\ A & \longrightarrow & X & \longrightarrow & R \end{array}$$

Recall that an i -cube in $C(A, X)$ can be represented by a pair of i -cubes in $C(A, R)$ and X that are mapped to the same i -cube in $S(\Gamma)$. Thus there exists a naturally embedded copy of A in $C(A, X)$ by considering $A \rightarrow C(A, R)$ and $A \rightarrow X \rightarrow R$. The component of $C(A, X)$ which contains this copy of A is called the *main component*. The canonical retraction is defined to be the composition $C(A, X) \rightarrow C(A, R) \rightarrow A$.

We recall the following notion from [19, Section 2.1]. It is also called a projection-like map in [39].

Definition 2.12. A cellular map between cube complexes is *cubical* if its restriction $\sigma \rightarrow \tau$ between cubes factors as $\sigma \rightarrow \eta \rightarrow \tau$, where the first map $\sigma \rightarrow \eta$ is a natural projection onto a face of σ and the second map $\eta \rightarrow \tau$ is an isometry.

Note that if the inverse image of each edge in $S(\Gamma)$ under $A \rightarrow S(\Gamma)$ is a disjoint union of vertices and single edges, then the retraction $C(A^1, R^1) \rightarrow A^1$ is cubical. Hence the canonical retraction $C(A, X) \rightarrow A$ is cubical. For example, the assumption is satisfied when X is directly special.

Definition 2.13. [39, Definition 3.14] Let X denote a cube complex. Let A and B be subcomplexes of X . We define $\text{WProj}_X(A \rightarrow B)$, the *wall projection* of A onto B in X , to be the union of B^0 together with all cubes of B whose edges are all parallel to edges of A .

Lemma 2.14. [39, Lemma 3.16] *Let A and D be locally convex subcomplex of a directed special cube complex B . Let \hat{D} denote the preimage of D in $\mathcal{C}(A, B)$, and let $r : \mathcal{C}(A, B) \rightarrow A$ be the canonical retraction map. Then $r(\hat{D}) \subset \text{WProj}_B(D \rightarrow A)$.*

Let A be a locally convex subcomplex in a special cube complex X . It is natural to ask what is the inverse image of A under the covering $\mathcal{C}(A, X) \rightarrow X$. This usually depends on how A sits inside X .

Definition 2.15. [39, Definition 3.10] Let $D \rightarrow C$ be a combinatorial map between cube complexes. Then a hyperplane of D is mapped to a hyperplane of C . Hence there is an induced map $V_D \rightarrow V_C$ between the set of hyperplanes in D and C respectively. The map $D \rightarrow C$ is *wall-injective* if $V_D \rightarrow V_C$ is injective.

Lemma 2.16. [39, Lemma 3.13] *Let X be a special cube complex and let $A \subset X$ be a wall-injective locally convex subcomplex. Then the preimage of A in $\mathcal{C}(A, X)$ is canonically isomorphic to $\mathcal{C}(A, A)$.*

Remark 2.17. Let A be a special cube complex. Usually $\mathcal{C}(A, A)$ is much larger than A . For example, let A be a circle made of n -edges for $n \geq 3$. Then $\mathcal{C}(A, A)$ is the disjoint union of a circle of length n and a circle of length $n(n-1)$.

Lemma 2.18. [39, Corollary 3.11] *Let $A \rightarrow X$ be a local-isometry. If the canonical retraction $\mathcal{C}(A, X) \rightarrow A$ is cubical, then A is wall-injective in $\mathcal{C}(A, X)$. In particular, A is wall-injective in $\mathcal{C}(A, X)$ when X is directly special.*

Definition 2.19. [39, Definition 3.17] Let $\pi : \bar{X} \rightarrow X$ denote a covering map. Let $A \subset X$ denote a connected subspace. An *elevation* of A to \bar{X} is a connected component of the preimage of A under $\bar{X} \rightarrow X$. If we choose base points $p \in A$ and $\bar{p} \in \bar{X}$ with $\pi(\bar{p}) = p$, then the *based elevation* of A is the component of the preimage of A that contains \bar{p} .

If $f : A \rightarrow X$ is any continuous map for A connected. We define an elevation \bar{A} of A to be a cover of A corresponding to the subgroup $f_*^{-1}(\pi_1(\bar{X}))$ (this depends on our choice of base points in A , X and \bar{X}). Then \bar{A} fits into the following commuting diagram:

$$\begin{array}{ccc} \bar{A} & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ A & \longrightarrow & X \end{array}$$

Remark 2.20. The above two notions of elevation are consistent in the following sense. Suppose $f : A \rightarrow X$ is continuous and let C be the mapping cylinder of f . Since C is homotopic to X , the covering map $\bar{X} \rightarrow X$ induces a covering map $\bar{C} \rightarrow C$. Then elevations of $A \rightarrow X$ corresponds to components in the inverse image of A in $\bar{C} \rightarrow C$.

Alternatively, let $\tilde{A} \rightarrow A$ be the pull-back of the $\bar{X} \rightarrow X$. Then an elevation of $A \rightarrow X$ corresponds to a connected component in \tilde{A} .

3. THE STRUCTURE OF BLOW-UP BUILDING

3.1. The blow-up building. We briefly review the Davis realization of right-angled buildings and the construction of blow-up buildings in [44]. The reader can find detailed proofs of the statements mentioned below in [44].

Let \mathcal{P} be the poset of standard flats in $X(\Gamma)$. Recall that an *interval* of \mathcal{P} is a subset of form $I_{a,b} = \{x \in \mathcal{P} \mid a \leq x \leq b\}$ for some $a, b \in \mathcal{P}$. Since every interval of \mathcal{P} is a Boolean lattice of finite rank, one can construct a cube complex $|\mathcal{B}|$ whose poset of cubes is isomorphic to poset of intervals in \mathcal{P} ([1, Proposition A.38]). $|\mathcal{B}|$ is called the *Davis realization of the right-angled building associated with $G(\Gamma)$* and it is a $CAT(0)$ cube complex [22].

More precisely, there is a 1-1 correspondence between k -cubes in $|\mathcal{B}|$ and intervals of form I_{F_1, F_2} where $F_1 \subset F_2$ are two standard flats in $X(\Gamma)$ with $\dim(F_2) - \dim(F_1) = k$. In particular, vertices of $|\mathcal{B}|$ correspond to standard flats in $X(\Gamma)$. We label each vertex of $|\mathcal{B}|$ by the support (Definition 2.2) of the corresponding standard flat (if a vertex of $|\mathcal{B}|$ corresponds to a 0-dimensional standard flat, then it is labelled by the empty set), and the *rank* of this vertex is defined to be the number of vertices in its label. Moreover, the vertex set of $|\mathcal{B}|$ inherits a partial order from \mathcal{P} .

We label each edge of $|\mathcal{B}|$ by the unique vertex of Γ in the symmetric difference of the labels of its two endpoints. Note that two parallel edges are labelled by the same vertex of Γ , hence there is an induced labelling of hyperplanes in $|\mathcal{B}|$. If two hyperplanes cross, then their labels are adjacent in Γ .

Let $K \subset X(\Gamma)$ be a Γ' -component and let $\mathcal{P}_K \subset \mathcal{P}$ be the sub-poset of standard flats inside K . Then \mathcal{P}_K gives rise to a convex subcomplex $|\mathcal{B}|_K \subset |\mathcal{B}|$. By definition, $|\mathcal{B}|_K$ is isomorphic to the Davis realization of the right-angled building associated with $G(\Gamma')$.

There is an induced action $G(\Gamma) \curvearrowright |\mathcal{B}|$ which preserves the labellings of vertices and edges. The action is cocompact, but not proper - the stabilizer of a cube is isomorphic to \mathbb{Z}^n where n is the rank of the minimal vertex in this cube. The following construction is motivated by the attempt to blow-up $|\mathcal{B}|$ with respect to this data of stabilizers (in a possibly non-equivariant way). See Theorem 3.11 for a precise statement.

Definition 3.1 (branched lines and flats). A *branched line* is constructed from a copy of \mathbb{R} with finitely many edges of length 1 attached to each integer point. We also require that the valence of each vertex in a branched line is bounded above by a uniform constant. This space has a natural simplicial structure and is endowed with the path metric. A *branched flat* is a product of finitely many branched lines.

Let β be a branched line. We call those vertices with valence = 1 in β the *tips* of β , and the collection of all tips is denoted by $t(\beta)$. The copy of \mathbb{R} in β is called the *core* of β . For a branched flat F , we define $t(F)$ to be the product of the tips of its factors, and the *core* of F to be the product of the cores of its factors.

Definition 3.2 (blow-up building). The following construction is a special case of [44, Section 5.2, Section 5.3]. Let $\mathcal{P}(\Gamma)$ be the extension complex. Pick a vertex $v \in \mathcal{P}(\Gamma)$, we associate v with a branched line β_v . Moreover, for each standard geodesic line ℓ with $\Delta(\ell) = v$ (see Definition 2.4), we associate a bijection $f_{v,\ell}$ from the 0-skeleton $\ell^{(0)}$ of ℓ to $t(\beta_v)$ such that if two standard lines ℓ, ℓ' are parallel, then $f_{v,\ell'} = f_{v,\ell} \circ p$, where $p : \ell'^{(0)} \rightarrow \ell^{(0)}$ is the map induced by parallelism. These $f_{v,\ell}$'s are called *blow-up data*.

We associate each standard flat $F \subset X(\Gamma)$ with a space β_F as follows. If F is a 0-dimensional standard flat (i.e. $\Delta(F) = \emptyset$), then β_F is a point. Suppose $\Delta(F) \neq \emptyset$.

Let $F = \prod_{v \in \Delta(F)} \ell_v$ be a product decomposition of F , here v is a vertex in $\Delta(F)$, and $\ell_v \subset F$ is a standard geodesic line with $\Delta(\ell_v) = v$. Then $\beta_F = \prod_{v \in \Delta(F)} \beta_v$.

For standard flats $F' \subset F$, suppose $F' = \prod_{v \in \Delta(F')} \ell_v \times \prod_{v \notin \Delta(F')} \{x_v\}$ (x_v is a vertex in ℓ_v). Then we define an isometric embedding $\beta_{F'} \hookrightarrow \beta_F$ as follows:

$$\beta_{F'} = \prod_{v \in \Delta(F')} \beta_v \cong \prod_{v \in \Delta(F')} \beta_v \times \prod_{v \notin \Delta(F')} \{f_{v, \ell_v}(x_v)\} \hookrightarrow \prod_{v \in \Delta(F)} \beta_v = \beta_F.$$

One readily verify that the above construction gives a functor from the poset \mathcal{P} to the category of branched flats with isometric embeddings as morphisms. Let $Y(\Gamma)$ be the space obtained by gluing the collection of all β_F 's according the isometric embeddings defined as above. The following properties follows from functorality of $F \rightarrow \beta_F$.

- (1) Each β_F is embedded in $Y(\Gamma)$. It is called a *standard branched flat*. Then core of a standard branched flat is called a *standard flat*. There is a 1-1 correspondence between standard flats in $X(\Gamma)$ and standard flats in $Y(\Gamma)$.
- (2) $\beta_{F_1} \cap \beta_{F_2} = \beta_{F_1 \cap F_2}$ [44, Lemma 8.1]. Thus if the cores of β_{F_1} and β_{F_2} have nontrivial intersection, then $\beta_{F_1} = \beta_{F_2}$. In particular, different standard flats in $Y(\Gamma)$ are disjoint.
- (3) We can glue all the $f_{v, \ell}$'s together to obtain an injective map $f : X(\Gamma)^{(0)} \rightarrow Y(\Gamma)$ such that f map the vertex set of each standard flat bijectively to the tips of a standard branched flat. The image of f is exactly the collection of 0-dimensional standard flats in $Y(\Gamma)$.

$Y(\Gamma)$ is called the *blow-up building of type Γ* and it is a $CAT(0)$ cube complex ([44]). We claim for each vertex $x \in Y(\Gamma)$, there is a unique standard flat containing x . Then this induces a map $\mathcal{L}_1 : Y(\Gamma)^{(0)} \rightarrow \mathcal{P}$ (recall that \mathcal{P} is the poset of standard flats in $X(\Gamma)$). The claim is clear when Γ is a clique (in this case $Y(\Gamma)$ is a product of branched lines). In general we can choose a standard branched flat containing x and find a standard flat inside which contains x . Since different standard flats in $Y(\Gamma)$ are disjoint, thus the uniqueness follows. We label each vertex $x \in Y(\Gamma)$ by the support of $\mathcal{L}_1(x)$. The *rank* of x is the number of vertices in this clique.

We record the following simple observation.

Lemma 3.3. *Pick standard flat $F \subset X(\Gamma)$, then for any vertex v in the core of β_F , we have $\mathcal{L}_1(v) = F$. Conversely, if a vertex $v \in Y(\Gamma)$ satisfies $\mathcal{L}_1(v) = F$, then v is in the core of β_F .*

Definition 3.4 (edge labelling and hyperplane labelling of $Y(\Gamma)$). We label each edge of $Y(\Gamma)$ by a vertex of Γ as follows. First we define a map \mathcal{L}_2 from the collection of edges of $Y(\Gamma)$ to vertices of $\mathcal{P}(\Gamma)$. Pick edge $e \subset Y(\Gamma)$, then e is contained in a standard branched flat β_F . There is a unique product factor β_v of β_F such that e is parallel to some edge in β_v . We define $\mathcal{L}_2(e) = v$. There may be several β_F 's that contain e , however they will give rise to the same v . \mathcal{L}_2 actually induces an edge-labelling of $Y(\Gamma)$ by vertices of Γ , since vertices of $\mathcal{P}(\Gamma)$ are labelled by the vertices of Γ (see Section 2.1).

Since every cube of $Y(\Gamma)$ is contained in some β_F , the opposite edges of a 2-cube are mapped to the same vertex under \mathcal{L}_2 . Thus $\mathcal{L}_2(e_1) = \mathcal{L}_2(e_2)$ if e_1 is parallel to e_2 . This induces labelling of hyperplanes in $Y(\Gamma)$ by vertices of Γ . Note that if two hyperplanes in $Y(\Gamma)$ cross, then their labels are adjacent in Γ .

It follows from Lemma 2.3 that each Γ' -component in $Y(\Gamma)$ or $|\mathcal{B}|$ is a convex subcomplex.

Example 3.5. Let Γ be a clique and $\{v_i\}_{i=1}^n$ be its vertex set. Then we can identify vertices of $\mathcal{P}(\Gamma)$ with vertices of Γ . In this case, $Y(\Gamma) \cong \prod_{i=1}^n \beta_{v_i}$ where β_{v_i} is a branched line. Let $p_i : Y(\Gamma) \rightarrow \beta_{v_i}$ be the projection map. For vertex $x \in Y(\Gamma)$, then v_i is in the label of x if and only if $p_i(x)$ is in the core of β_{v_i} . For edge $e \subset Y(\Gamma)$, the label of e is the unique v_i such that $p_i(e)$ is an edge.

Let $e \subset Y(\Gamma)$ be an edge and v_1, v_2 be its endpoints. If v_1 and v_2 have the same rank, then $\mathcal{L}_1(v_1) = \mathcal{L}_1(v_2)$, hence they are labelled by the same clique in Γ , and the label of e belongs to this clique. Otherwise, one of $\mathcal{L}_1(v_1)$ and $\mathcal{L}_1(v_2)$ is a codimension one flat in another. Thus v_1 and v_2 are labelled by two cliques such that one is contained in another. The label of e is the unique vertex in the difference of these two cliques. Again, to prove these statements, it suffices to consider the case when Γ is a clique. The general case reduces to this special case by considering a standard branched flat containing e . Moreover, we have the following consequence.

Lemma 3.6.

- (1) *Pick an edge e in $Y(\Gamma)$ (or $|\mathcal{B}|$). Then the label of e is contained in the label of at least one of its endpoints.*
- (2) *Pick an edge path ω connecting vertices x and y in $Y(\Gamma)$ (or $|\mathcal{B}|$). Then for every vertex in the symmetric difference of the labellings of x and y , there exists an edge in ω labelled by this vertex.*

Definition 3.7. Pick a Γ' -component $K \subset X(\Gamma)$. We define $\beta_K = \cup_F \beta_F$ with F ranging over standard flats in K .

We recall the following properties of β_K , where (1) follows from [44, Lemma 5.18] and (2) follows from [44, Corollary 5.16].

Lemma 3.8.

- (1) *The subcomplex β_K is convex in $Y(\Gamma)$, and it is the convex hull of $f(K^{(0)})$ (f is introduced in Definition 3.2).*
- (2) *Suppose Γ' admits a join decomposition $\Gamma' = \Gamma'_1 \circ \Gamma'_2$. For $i = 1, 2$, pick Γ'_i -component $K_i \subset K$ and let $\pi_i : \beta_K \rightarrow \beta_{K_i}$ be the $CAT(0)$ projection ([16, Proposition II.2.4]). Then $\pi_1 \times \pi_2 : \beta_K \rightarrow \beta_{K_1} \times \beta_{K_2}$ is a cubical isomorphism. Similarly, $|\mathcal{B}|_K \cong |\mathcal{B}|_{K_1} \times |\mathcal{B}|_{K_2}$.*

Lemma 3.9. *Pick vertex $v \in \mathcal{P}(\Gamma)$. Let P_v be as in Definition 2.5.*

- (1) *The subcomplex β_{P_v} admits a natural $CAT(0)$ cubical product decomposition*
- $$(3.10) \quad \beta_{P_v} \cong \beta_v \times \beta_v^\perp.$$
- (2) *Let $e \subset Y(\Gamma)$ be an edge such that $v = \mathcal{L}_2(e)$. Then the hyperplane h_e dual to e can be identified as the product factor β_v^\perp in (3.10).*

Proof. Suppose v is labelled by $\bar{v} \in \Gamma$. Note that P_v is a $St(\bar{v})$ -component in $X(\Gamma)$. By construction, any standard $\{\bar{v}\}$ -component in β_{P_v} can be naturally identified with β_v . Let $\beta_{P_v} \cong \beta_v \times \beta_v^\perp$ be the product decomposition induced by the join decomposition $St(\bar{v}) = \{\bar{v}\} \circ lk(\bar{v})$ as in Lemma 3.8 (2).

To see (2), recall that if e' is parallel to e , then $\mathcal{L}_2(e') = \mathcal{L}_2(e)$. Thus it suffices to prove if an edge $e' \subset Y(\Gamma)$ satisfies $\mathcal{L}(e') = v$, then $e' \subset \beta_{P_v}$. To see this, note that by the definition of \mathcal{L}_2 , there exists standard flat $F \subset X(\Gamma)$ such that $e' \subset \beta_F$ and $v \in \Delta(F)$. Thus $F \subset P_v$ and $e' \subset \beta_{P_v}$. \square

3.2. Canonical projection and standard components. If we collapse each standard flat in $Y(\Gamma)$, then the resulting space is $|\mathcal{B}|$. More precisely, we define the *canonical projection* $\pi : Y(\Gamma) \rightarrow |\mathcal{B}|$ as follows. Let $\mathcal{L}_1 : Y(\Gamma)^{(0)} \rightarrow \mathcal{P} \cong |\mathcal{B}|^{(0)}$ be as in Definition 3.2. We claim \mathcal{L}_1 can be extended to a cubical map, which defines π (such extension, if exists, must be unique). When Γ is a clique, $Y(\Gamma)$ is a product of branched line, then it is clear that \mathcal{L}_1 can be extended to a map which collapses the core of each factors. In general, for each standard flat $F \subset X(\Gamma)$, we can extend $\mathcal{L}_1|_{\beta_F^{(0)}}$ to a map $\pi_F : \beta_F \rightarrow |\mathcal{B}|_F$. The uniqueness of extension implies we can piece together these π_F 's to define π . By definition, π preserves the rank and labelling of vertices, and it induces a bijection between vertices of rank 0 in $Y(\Gamma)$ and $|\mathcal{B}|$.

Pick an edge $e \subset Y(\Gamma)$, e is *vertical* if $\pi(e)$ is a point, otherwise e is *horizontal*. In the latter case, e and $\pi(e)$ have the same label. Note that e is vertical if and only if its two endpoints of e have the same rank. Edges parallel to a vertical (or horizontal) edge are also vertical (or horizontal), thus it makes sense to talk about vertical (or horizontal) hyperplanes. It is shown in [44] that π is a restriction quotient in the sense that if one start with the wall space associated with $Y(\Gamma)$ and pass to a new wall space by forgetting all the vertical hyperplanes, then the new dual cube complex is $|\mathcal{B}|$. The following is a consequence of Theorem 4.4, Corollary 5.15 and the discussion in the beginning of Section 5.2 of [44].

Theorem 3.11. *For any cube $\sigma \subset |\mathcal{B}|$ and an interior point $x \in \sigma$, $\pi^{-1}(x)$ is isomorphic to \mathbb{E}^n where n is the rank of the minimal vertex in σ . Conversely, if Y is any $CAT(0)$ cube complex such that there exists a cubical map $\pi : Y \rightarrow |\mathcal{B}|$ which satisfies the property in the previous sentence, then Y can be constructed via Definition 3.2.*

Definition 3.12. A *standard Γ' -component* of $Y(\Gamma)$ (or $|\mathcal{B}|$) is a Γ' -component which contains a vertex of rank 0.

Lemma 3.13. *Let K be a Γ' -component in $|\mathcal{B}|$ (or $Y(\Gamma)$). Then the following are equivalent.*

- (1) K is standard.
- (2) There exists one vertex in K whose label is contained in Γ' .
- (3) The label of each vertex in K is contained in Γ' .

Proof. (1) \Rightarrow (2) is trivial (K contains a vertex of rank 0). (2) \Rightarrow (3) follows from Lemma 3.6 (2) and the connectedness of K . For (3) \Rightarrow (1), pick vertex $x \in K$ labelled by $\Gamma_x \subset \Gamma'$ and let β_F be the minimal standard branched flat containing x . Then x is in the core of β_F by minimality. Then the discussion in Example 3.5 implies the label of each edge of β_F is in Γ_x . Thus $\beta_F \subset K$ and in particular K contains a rank 0 vertex. \square

Lemma 3.14. *The map π induces a one to one correspondence between standard Γ' -components in $|\mathcal{B}|$ and $Y(\Gamma)$ in the following sense.*

- (1) *If $K \subset |\mathcal{B}|$ is a standard Γ' -component, then $\pi^{-1}(K)$ is a standard Γ' -component.*
- (2) *If $L \subset Y(\Gamma)$ is a standard Γ' -component, then $\pi(L)$ is a standard Γ' -component.*

Proof. We prove (1). It follows from [44, Theorem 4.4] that $\pi^{-1}(K)$ is convex, in particular connected. Now we verify Definition 2.2 (2). Suppose there exists an

edge in $\pi^{-1}(K)$ labelled by $v \notin \Gamma'$. Then there is an endpoint of this edge labelled by a clique $C \subset \Gamma$ such that $v \in C$. Let $x \in K$ be the π -image of this endpoint. Then x is also labelled by C . Pick a rank 0 vertex $y \in K$. Then the connectedness of K implies we can join x and y by an edge path $\omega \subset K$. Since y is labelled by the empty set, Lemma 3.6 (2) implies at least one edge in ω is labelled by v , which is a contradiction. The second part of Definition 2.2 (2) is clear. The maximality of $\pi^{-1}(K)$ follows from the maximality of K .

To see (2), note that each edge of $\pi(L)$ is labelled by a vertex in Γ' , and $\pi(L)$ contains a rank 0 vertex. Let K' be the standard Γ' -component containing $\pi(L)$. By (1), $\pi^{-1}(K')$ is a standard Γ' -component containing L , thus $L = \pi^{-1}(K')$ and $K' = \pi(L)$. \square

Lemma 3.15. *For each Γ' -component $K \subset X(\Gamma)$, $|\mathcal{B}|_K$ and β_K are standard Γ' -components. Conversely, each standard Γ' -component in $|\mathcal{B}|$ or $Y(\Gamma)$ must arise in such a way. In particular, each vertex of rank 0 is contained in some standard Γ' -component.*

Proof. We prove the above statement in $Y(\Gamma)$, the case of $|\mathcal{B}|$ is similar. The definition of β_K implies each edge of β_K is labelled by a vertex in Γ' , thus there exists a standard Γ' -component $L \subset Y(\Gamma)$ containing β_K . Let $\omega \subset L$ be an edge path from a vertex in β_F to an arbitrary vertex in L and let $\{v_i\}_{i=1}^n$ be consecutive vertices in ω . Let $F_i = \mathcal{L}_1(v_i)$ (see Definition 3.2 for the map \mathcal{L}_1). Then $F_i \cap F_{i+1} \neq \emptyset$ and the support of each F_i is in Γ' (by Lemma 3.13). So all F_i 's are in the same Γ' -component of $X(\Gamma)$. Thus for each vertex $v \in L$, $F_v = \mathcal{L}_1(v) \subset K$. It follows from Lemma 3.3 that $v \in \beta_{F_v} \subset \beta_K$. Thus $\beta_K = L$. To see the converse, pick a rank 0 vertex x in an arbitrary standard Γ' -component L' . Let $K' \subset X(\Gamma)$ be the Γ' -component containing $\mathcal{L}_1(x)$. Then $\beta_{K'} \cap L' \neq \emptyset$. Since $\beta_{K'}$ and L' are both standard Γ' -components, they are the same complex. \square

Lemma 3.16. *For $i = 1, 2$, let L_i be a standard Γ_i -component in $|\mathcal{B}|$ (or $Y(\Gamma)$) and let $K_i \subset X(\Gamma)$ be the associated Γ_i -component. Then $L_1 \cap L_2 \neq \emptyset$ if and only if $K_1 \cap K_2 \neq \emptyset$. In this case, $L_1 \cap L_2$ is a standard $\Gamma_1 \cap \Gamma_2$ -component.*

Proof. We prove the lemma in $Y(\Gamma)$, the case of $|\mathcal{B}|$ is similar. It is clear from the definitions that $K_1 \cap K_2 \neq \emptyset$ implies $L_1 \cap L_2 \neq \emptyset$. To see the converse, pick vertex $x \in L_1 \cap L_2$, then the argument in Lemma 3.13 implies that the minimal standard branched flat containing x is inside both L_1 and L_2 . In particular, $L_1 \cap L_2$ contains a vertex of rank 0, which implies $K_1 \cap K_2 \neq \emptyset$. By Lemma 3.15, there exists a standard $\Gamma_1 \cap \Gamma_2$ -component N containing this rank 0 vertex. It is clear that $N \subset L_1 \cap L_2$ and $L_1 \cap L_2 \subset N$. \square

Let L be a (not necessarily standard) Γ_L -component in $Y(\Gamma)$ and let $x \in L$ be a vertex. Suppose Γ_L^\perp is the clique spanned by vertices in $\Gamma_x \setminus \Gamma_L$, where Γ_x is the label of x . Then Lemma 3.6 (2) implies Γ_L^\perp is contained in the label of each vertex of L , and Lemma 3.6 (1) implies Γ contains the graph join $\Gamma_L \circ \Gamma_L^\perp$.

The proof of Lemma 3.13 implies x is connected to a rank 0 vertex x_0 via an edge-path such that the label of each edge in this path is contained in Γ_L^\perp . Let M be the $\Gamma_L \circ \Gamma_L^\perp$ -component containing x_0 and let $M = N \times N^\perp$ be the product decomposition in Lemma 3.8 (2). Note that $L \subset M$ and L has trivial projection to N^\perp . Thus L must be of form $\{y\} \times N$ for some vertex $y \in N^\perp$, and L is standard

if and only if y is of rank 0. We can characterize standard Γ' -components in $|\mathcal{B}|$ in a similar way.

Lemma 3.17.

- (1) Any Γ' -component in $Y(\Gamma)$ or $|\mathcal{B}|$ is parallel to (hence isomorphic to) a standard Γ' -component.
- (2) If Γ' is a subgraph satisfying $\Gamma' \not\subseteq \text{lk}(v)$ for any vertex $v \in \Gamma \setminus \Gamma'$, then each Γ' -component is standard.
- (3) If e_1 and e_2 are two edges in $Y(\Gamma)$ or $|\mathcal{B}|$ emanating from the same vertex, then they span a square if and only if their labels are adjacent in Γ .

Proof. It remains to prove the if direction of (3). Let L be the Γ' -component containing e_1 and e_2 where Γ' is the edge spanned by the label of e_1 and e_2 . Then (1) implies L is a product of two branched lines in the case of $Y(\Gamma)$, and is a product of two infinite trees of diameter 2 in the case of $|\mathcal{B}|$, thus (3) follows. Alternatively, one can also prove (3) directly by looking at the link of vertices, see [44, Section 5.4]. \square

4. A GEOMETRIC MODEL FOR GROUPS Q.I. TO RAAGS

In [44], it was shown that if a group H is quasi-isometric to $G(\Gamma)$ such that $\text{Out}(G(\Gamma))$ is finite, then H acts geometrically on a $CAT(0)$ cube complex which is very similar to $X(\Gamma)$. In this section, we briefly recall the construction in [44], and prove several further properties of this construction.

4.1. Quasi-action on RAAG's. We pick an identification between $G(\Gamma)$ and the 0-skeleton $X^{(0)}(\Gamma)$ of $X(\Gamma)$. It follows from [41] that quasi-isometries between RAAG's with finite outer-automorphism group are at bounded distance from a canonical representative that respects standard flats, in the following sense.

Definition 4.1. A quasi-isometry $\phi : X^{(0)}(\Gamma) \rightarrow X^{(0)}(\Gamma)$ is *flat-preserving* if it is a bijection and for every standard flat $F \subset X(\Gamma)$ there is a standard flat $F' \subset X(\Gamma)$ such that ϕ maps the 0-skeleton of F bijectively onto the 0-skeleton of F' . The standard flat F' is uniquely determined, and we denote it by $\phi_*(F)$.

Theorem 4.2 (Vertex rigidity [41, 9]). *Suppose $\text{Out}(G(\Gamma))$ is finite and $G(\Gamma) \not\cong \mathbb{Z}$. Let $\phi : X^{(0)}(\Gamma) \rightarrow X^{(0)}(\Gamma)$ be an (L, A) -quasi-isometry. Then there is a unique flat-preserving quasi-isometry $\bar{\phi} : X^{(0)}(\Gamma) \rightarrow X^{(0)}(\Gamma)$ at finite distance from ϕ , and moreover*

$$d(\bar{\phi}, \phi) = \sup\{v \in X^{(0)}(\Gamma) \mid d(\bar{\phi}(v), \phi(v))\} < D = D(L, A).$$

Given any quasi-action $H \curvearrowright^{\rho} X^{(0)}(\Gamma)$, we may apply Theorem 4.2 to the associated family of quasi-isometries $\{\rho(h) : X^{(0)}(\Gamma) \rightarrow X^{(0)}(\Gamma)\}_{h \in H}$ to obtain a new quasi-action $H \curvearrowright^{\bar{\rho}} X^{(0)}(\Gamma)$ by flat-preserving quasi-isometries. Due to the uniqueness assertion in Theorem 4.2, the quasi-action $\bar{\rho}$ is actually an action, i.e. the composition law $\bar{\rho}(hh') = \bar{\rho}(h) \circ \bar{\rho}(h')$ holds exactly, not just up to bounded error.

Pick vertex $v \in \mathcal{P}(\Gamma)$ and let P_v be the v -parallel set (Definition 2.5). Note that there is a cubical product decomposition $P_v \simeq \mathbb{R}_v \times Q_v$, where \mathbb{R}_v is parallel to some standard geodesic ℓ with $\Delta(\ell) = v$. Likewise, there is a product decomposition of the 0-skeleton

$$(4.3) \quad P_v^{(0)} \simeq \mathbb{Z}_v \times Q_v^{(0)}$$

where \mathbb{Z}_v is the 0-skeleton of \mathbb{R}_v equipped with the induced metric.

Now let $\rho : H \curvearrowright X^{(0)}$ be an action by flat-preserving (L, A) -quasi-isometries. It follows readily from Definition 4.1 that the action respects parallelism of standard geodesics and standard flats, and this implies there is an induced action of H on $\mathcal{P}(\Gamma)$. For each vertex $v \in \mathcal{P}(\Gamma)$, let $H_v \leq H$ be the stabilizer of v . The action $H_v \curvearrowright P_v^{(0)}$ preserves the product decomposition (4.3), and therefore gives rise to a factor action $\rho_v : H_v \curvearrowright \mathbb{Z}_v$ by (L, A) -quasi-isometries.

Theorem 4.4. [44, Corollary 6.17] *Let $\rho : H \curvearrowright G(\Gamma)$ be an action by flat-preserving bijections. Suppose*

- (1) *the induced action $H \curvearrowright \mathcal{P}(\Gamma)$ preserves the label of vertices in $\mathcal{P}(\Gamma)$;*
- (2) *for each vertex $v \in \mathcal{P}(\Gamma)$, the action $\rho_v : H_v \curvearrowright \mathbb{Z}$ is conjugate to an action by translations.*

Then the action ρ is conjugate to an action $H \curvearrowright G(\Gamma)$ by left translations via a flat-preserving bijection.

Now we impose a condition which guarantees assumption (1) of Theorem 4.4.

Definition 4.5. A finite simplicial graph Γ is *star-rigid* if for any automorphism α of Γ , α is identity whenever it fixes a closed star of some vertex $v \in \Gamma$ point-wise.

Lemma 4.6. *Suppose Γ is star-rigid, and $\text{Out}(G(\Gamma))$ is finite. If H acts on $\mathcal{P}(\Gamma)$ by simplicial isomorphisms, then there exists a finite index subgroup $H' \leq H$ such that the induced action $H' \curvearrowright \mathcal{P}(\Gamma)$ is label-preserving.*

Proof. Since $\text{Out}(G(\Gamma))$ is finite, there is an induced action $H \curvearrowright G(\Gamma)$ by flat-preserving maps. Let $F(\Gamma)$ be the flag complex of Γ . We define a homomorphism $\alpha : H \rightarrow \text{Aut}(F(\Gamma))$ as follows.

Each vertex $x \in X(\Gamma)$ gives rise to a simplicial embedding $i_x : F(\Gamma) \rightarrow \mathcal{P}(\Gamma)$ by considering the collection of standard flats passing through x . Pick $h \in H$ and vertex $x \in X(\Gamma)$, we define $\alpha(h, x) = i_{h(x)}^{-1} \circ h \circ i_x$. We claim $\alpha(h, x)$ does not depend on the choice of x . It suffices to show $\alpha(h, x') = \alpha(h, y')$ for any two vertices x' and y' in the same standard geodesic ℓ . Let $v = i_{x'}^{-1}(\Delta(\ell))$. Then $\alpha(h, x')$ and $\alpha(h, y')$ coincident on the closed star of v , hence they are equal. Now there is a well-defined map $\alpha : H \rightarrow \text{Aut}(F(\Gamma))$. One readily verify that α is a group homomorphism and we take $H' = \ker(\alpha)$. \square

Assumption (2) of Theorem 4.4 is not true in general, see Proposition 4.8. However, the following weaker version holds under additional conditions. The proof is postponed to the next section.

Lemma 4.7. *Let $\rho : H \curvearrowright G(\Gamma)$ be an action by flat-preserving bijections which are (L, A) -quasi-isometries. Suppose the action is cobounded and discrete. Then for each vertex $v \in \mathcal{P}(\Gamma)$, there exists a subgroup $H'_v \leq H_v$ of finite index such that the action $\rho_v : H_v \curvearrowright \mathbb{Z}_v$ restricted on H'_v is conjugate to an action by translations.*

The following is the key to understand the factor actions $\rho_v : H_v \curvearrowright \mathbb{Z}_v$.

Proposition 4.8. [44, Proposition 6.2] *If $U \xrightarrow{\rho_0} \mathbb{Z}$ is an action by (L, A) -quasi-isometries, then there exist a branched line β without valence 2 vertices, an action $U \xrightarrow{\rho_1} \beta$ by simplicial isomorphisms and a bijective equivariant (L', A') -quasi-isometry (here L', A' depends only on L and A)*

$$U \xrightarrow{\rho_0} \mathbb{Z} \longrightarrow U \xrightarrow{\rho_1} t(\beta).$$

Note that the valence of each vertex in β is uniformly bounded above by some constant depending only on L and A .

Definition 4.9. (an equivariant construction, [44, Section 5.6 and Section 6.1]) Let $\rho : H \curvearrowright G(\Gamma)$ be an action by flat-preserving bijections which are also (L, A) -quasi-isometries. We will use Definition 3.2 to construct an isometric action $\rho' : H \curvearrowright Y(\Gamma)$ from ρ . First we need to choose the blow-up data $f_{v,\ell}$'s which are compatible with the H -action. Actually, it suffices to associate a branched line β_v and a map $f_v : \mathbb{Z}_v \rightarrow t(\beta_v)$ to each vertex $v \in \mathcal{P}(\Gamma)$, since for each standard geodesic ℓ such that $\Delta(\ell) = v$, we can identify vertices of ℓ with \mathbb{Z}_v via (4.3), which induces the map $f_{v,\ell}$. To define f_v 's, we start with the factor actions $\rho_v : H_v \curvearrowright \mathbb{Z}_v$. Note that there exists $L_1, A_1 > 0$ such that each ρ_v is an action by (L_1, A_1) -quasi-isometries.

Consider the induced action $H \curvearrowright \mathcal{P}(\Gamma)$ and pick a representative from each H -orbit of vertices of $\mathcal{P}(\Gamma)$. The resulting set is denoted by V . For each $v \in V$, by Proposition 4.8, there exist a branched line β_v , an isometric action $H_v \curvearrowright \beta_v$ and an H_v -equivariant bijection $f_v : H_v \curvearrowright \mathbb{Z}_v \rightarrow H_v \curvearrowright t(\beta_v)$. Moreover, the valence of vertices in each β_v is uniformly bounded from above in terms of L_1 and A_1 . For vertex $v \notin V$, we pick an element $h \in H$ such that $h(v) \in V$. Note that h induces a bijection $h' : \mathbb{Z}_v \rightarrow \mathbb{Z}_{h(v)}$. We define $\beta_v = \beta_{h(v)}$ and $f_v = f_{h(v)} \circ h' : \mathbb{Z}_v \rightarrow t(\beta_v)$.

Let $Y(\Gamma)$ be the associated blow-up building and let $f : G(\Gamma) \rightarrow Y(\Gamma)$ be the map as in Definition 3.2. The following are true (see [44, Theorem 6.4]).

- (1) The complex $Y(\Gamma)$ is uniformly locally finite (this essentially follows from the fact the complexity of all β_v 's are uniformly bounded from above).
- (2) The bijections $f_{v,\ell}$'s and the action ρ induce an action $\rho' : H \curvearrowright Y(\Gamma)$ by cubical isomorphisms. For each element $h \in H$, $\rho'(h)$ maps standard branched flats to standard branched flats. Hence $\rho'(h)$ also preserves the collection of β_{P_v} 's (see Lemma 3.9 and Definition 3.7 for β_{P_v}).
- (3) The map f is an H -equivariant quasi-isometry.
- (4) If ρ is discrete and cobounded, then ρ' is geometric, i.e. the action of ρ' on $Y(\Gamma)$ is proper and cocompact.

4.2. Stabilizer of parallel sets. In this section we restrict ourselves to the case when $\rho : H \curvearrowright G(\Gamma)$ is a proper and cobounded action by (L, A) -quasi-isometries which are also flat-preserving bijections. In this case, the action $\rho' : H \curvearrowright Y(\Gamma)$ constructed in Definition 4.9 is geometric. Since the collection $\{\beta_{P_v}\}_{v \in \text{Vertex}(\mathcal{P}(\Gamma))}$ is H -invariant and locally finite, the action $\text{Stab}_H(\beta_{P_v}) \curvearrowright \beta_{P_v}$ is cocompact. However, $H_v \leq \text{Stab}_H(\beta_{P_v})$ is of finite index. Thus the action $H_v \curvearrowright \beta_{P_v}$ is geometric. Note that $H_v \curvearrowright \beta_{P_v}$ respects the product decomposition $\beta_{P_v} \simeq \beta_v \times \beta_v^\perp$ (Lemma 3.9), hence there is an induced factor action $H_v \curvearrowright \beta_v$, which is cobounded. If H_v contains element that flips the two ends of β_v , then β_v/H_v is a tree. Otherwise H_v acts by translations on the core of β_v , which gives rise to a homomorphism $\phi : H_v \rightarrow \mathbb{Z}$ by associating each element in H_v its translation length. Let K_v be the kernel of ϕ . Then we have a splitting exact sequence $1 \rightarrow K_v \rightarrow H_v \rightarrow \mathbb{Z} \rightarrow 1$. Let $\alpha \in H_v$ be a lift of a generator of \mathbb{Z} .

Lemma 4.10. *Suppose $\rho : H \curvearrowright G(\Gamma)$ is discrete and cobounded and suppose H_v does not flip the two ends of β_v . Then the conjugation map $K_v \rightarrow \alpha K_v \alpha^{-1}$ gives rise to an element of finite order in $\text{Out}(K_v)$. Thus there exists integer $n > 0$ such that $\phi^{-1}(n\mathbb{Z})$ is isomorphic to $K_v \oplus n\mathbb{Z}$. Moreover, up to passing to a possibly larger n , we can choose the $n\mathbb{Z}$ factor such that it acts trivially on β_v^\perp .*

Proof. Since H_v preserves the product structure of β_{P_v} , we have a homomorphism $h : H_v \rightarrow \text{Aut}(\beta_v^\perp)$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} K_v & \xrightarrow{C_\alpha} & K_v \\ h \downarrow & & \downarrow h \\ h(K_v) & \xrightarrow{C_{h(\alpha)}} & h(K_v) \end{array}$$

Here C_α and $C_{h(\alpha)}$ denote conjugation by α and $h(\alpha)$ respectively. Note that the action $h(K_v) \curvearrowright \beta_v^\perp$ is geometric. We apply Lemma 4.11 below to the vertex set of β_v^\perp to deduce that $h(K_v)$ is of finite index in $h(H_v)$. Thus $C_{h(\alpha)}$ gives rise to a finite order element in $\text{Out}(h(K_v))$. Since $h : K_v \rightarrow h(K_v)$ has finite kernel, C_α is of finite order in $\text{Out}(K_v)$. Then we can find integer $n > 0$ such that $\phi^{-1}(n\mathbb{Z}) = K_v \oplus \langle n\alpha \rangle$. Apply Lemma 4.11 to the vertex set of β_v^\perp again, we know there exists integer $m > 0$ such that $h(mn\alpha) \in h(K_v)$. Suppose $h(mn\alpha) = h(\beta)$ for $\beta \in K_v$. Then $\phi^{-1}(mn\mathbb{Z}) = K_v \oplus \langle (mn\alpha)\beta^{-1} \rangle$, where $(mn\alpha)\beta^{-1}$ acts trivially on β_v^\perp . \square

Lemma 4.11 (B. Kleiner). *Suppose Z is a metric space such that every r -ball contains at most $N = N(r)$ elements. Assume that $A \curvearrowright Z$ is a faithful action by (L, A) -quasi-isometries, and B is a normal subgroup of A that acts discretely cocompactly on Z . Then B is of finite index in A .*

Proof. Suppose B is of infinite index in A . Let $\{\alpha_i\}_{i \in I} \subset A$ be a collection of elements such that different α_i 's are in different B -cosets. Pick a base point $p \in Z$. Since the B -action is cocompact, we can assume there exists $D > 0$ such that $d(\alpha_i(p), p) < D$ for all $i \in I$.

Note that B is finitely generated from our assumption. Let $\{b_\lambda\}_{\lambda \in \Lambda}$ be a finite generating set for B . For each b_λ , there exists $D' > 0$ such that $d(\alpha_i b_\lambda \alpha_i^{-1}(p), p) < D'$ for all $i \in I$. Since $\alpha_i b_\lambda \alpha_i^{-1} \in B$, by the discreteness of B -action, we can assume without loss of generality that $\alpha_i b_\lambda \alpha_i^{-1} = \alpha_j b_\lambda \alpha_j^{-1}$ for any $i \neq j$. Since there are only finitely many b_λ 's, we can also assume the equality holds for all b_λ 's. Thus $\alpha_i b \alpha_i^{-1} = \alpha_j b \alpha_j^{-1}$ for any $b \in B$ and $i, j \in I$. We can further assume without loss of generality that $\alpha_i b \alpha_i^{-1} = b$ for any $i \in I$ and $b \in B$.

Since α_i commutes with each element of B , by the cocompactness of B -action, there exists $R > 0$ such that α_i is completely determined by its behavior on the R -ball centred at p . However, $d(\alpha_i(p), p) < D$ for all $i \in I$, so there are only finitely many ways to define α_i . Thus there exists a pair $i \neq j$ such that $\alpha_i = \alpha_j$, which yields a contradiction. \square

Proof of Lemma 4.7. Up to passing to a subgroup of index 2, we assume H_v does not flip the ends of β_v . Let $h : H_v \rightarrow \text{Isom}(\beta_v)$ be the homomorphism induced by the action $H_v \curvearrowright \beta_v$ and let $\bar{H}_v = \text{Im } h$. We claim the action $\bar{H}_v \curvearrowright \beta_v$ is geometric. Assuming the claim, we deduce that \bar{H}_v has a finite index subgroup A isomorphic to \mathbb{Z} . Define $H'_v = h^{-1}(A)$ and the lemma follows.

It suffices to show the subgroup of \bar{H}_v which stabilizes a tip of β_v is finite. Pick $x \in t(\beta_v)$. Then the subgroup $K \leq H_v$ which stabilizes $\{x\} \times \beta^\perp$ acts geometrically on itself, thus K is finitely generated. Since H_v does not flip the ends of β_v , K acts trivially on the core of β_v . Thus $h(K)$ is a finitely generated subgroup of $\prod_{i \in I} G_i$, here each G_i is the permutation group of n_i points and there is a uniform upper bound for all the n_i 's. Hence $h(K)$ is finite. \square

4.3. Special cube complexes and blow-up buildings. Suppose Γ is star-rigid and $\text{Out}(G(\Gamma))$ is finite. Let H be a finitely generated group quasi-isometric to $G(\Gamma)$ and let $\rho : H \curvearrowright G(\Gamma)$ be the associated action by flat-preserving maps. By Lemma 4.6, we can assume the induced action $H \curvearrowright \mathcal{P}(\Gamma)$ preserves the labelling of vertices (up to passing to a finite index subgroup). Then Definition 4.9 gives rise to a geometric action of H on a blow-up building $Y(\Gamma)$, moreover, this action preserves the labelling of edges.

Lemma 4.12. *Suppose there exists a torsion free subgroup $H' \leq H$ of finite index. Then the following are equivalent:*

- (1) *For each vertex $v \in \mathcal{P}(\Gamma)$, the restricted action $\rho_v : H'_v \curvearrowright \mathbb{Z}_v$ is conjugate to an action by translations ($H'_v \leq H'$ is the stabilizer of v).*
- (2) *$Y(\Gamma)/H'$ is a special cube complex.*

If either (1) or (2) is true, then H' is isomorphic to a finite index subgroup of $G(\Gamma)$, hence H is commensurable to $G(\Gamma)$.

Proof. (1) \Rightarrow (2): Let $H'_v \curvearrowright \beta_v$ be the restricted action on the associated branched line, then (1) implies the image of $H'_v \rightarrow \text{Isom}(\beta_v)$ is isomorphic to \mathbb{Z} . Thus there exists an H'_v -invariant orientation of edges such that for any two adjacent edges of β_v which are in the same H'_v orbit, the orientation on them avoid following two configurations:



For each H' -orbit of vertices in $\mathcal{P}(\Gamma)$, we pick a representative v and assign an H'_v -equivariant orientation of edges in β_v which satisfies the above condition. This induces orientation of edges in β_{P_v} which are parallel to the β_v factor (see (3.10)). The H' -orbits of these oriented edges cover the 1-skeleton of $Y(\Gamma)$ (see (2) of Lemma 3.9), thus we have an H' -invariant edge orientation of $Y(\Gamma)$. This orientation respects parallelism of edges, thus each hyperplane of $Y(\Gamma)/H'$ is 2-sided. Since H' preserves the edge-labelling of $Y(\Gamma)$, hyperplanes of $Y(\Gamma)/H'$ do not self-intersect. Inter-osculation is ruled out by Lemma 3.17 (3). The above condition for orientation of edges of β_v implies hyperplanes of $Y(\Gamma)/H'$ do not directly self-osculate.

(2) \Rightarrow (1): Let \bar{H}'_v be the image of $H'_v \rightarrow \text{Isom}(\beta_v)$. It suffices to show \bar{H}'_v is isomorphic to \mathbb{Z} . The proof of Lemma 4.7 implies \bar{H}'_v has a finite index subgroup isomorphic to \mathbb{Z} . Thus it suffices to show \bar{H}'_v is torsion-free. Suppose the contrary is true, either there is an element of \bar{H}'_v that flips the two endpoints of an edge in β_v , which gives rise to a 1-sided hyperplane in $Y(\Gamma)/H'$, or there exists an element of \bar{H}'_v that permutes edges in the closed star of some vertex of β_v in a non-trivial way, which gives rise to a directly self-osculating hyperplane in $Y(\Gamma)/H'$.

The last statement of the lemma follows from Theorem 4.4. \square

5. BRANCHED COMPLEXES WITH TRIVIAL HOLONOMY

In this section, we look at more properties of the quotient of a blow-up building by a proper and cocompact group action. In particular, we introduce a notion of trivial holonomy for such quotient, under which condition one can collapse tori in the quotient to obtain another special cube complex.

5.1. The branched complex. Let $Y(\Gamma)$ be a blow-up building as in Definition 3.2 and let $|\mathcal{B}|$ be the Davis realization of the right-angled building associated with $G(\Gamma)$. In the light of Section 4.3, we assume H acts geometrically on $Y(\Gamma)$ by label-preserving cubical isomorphisms. A *branched complex of type Γ* is the orbifold $K(\Gamma) = Y(\Gamma)/H$. It is *torsion free* if H is torsion free (in this case the action $H \curvearrowright Y(\Gamma)$ is free), and it is *special* if H is torsion free and $K(\Gamma)$ is a special cube complex. If H is torsion free, and Γ is a clique or a point, then the corresponding branched complex is also called a *branched torus* or a *branched circle*. The *core* of a branched torus is the quotient of the core of $Y(\Gamma)$.

Lemma 5.1. *H preserves the rank and label of vertices in $Y(\Gamma)$.*

Proof. The discussion before Lemma 3.6 implies for any edge $e \subset Y(\Gamma)$ with its endpoints denoted by v_1 and v_2 , one can deduce the label of v_2 from the label of v_1 , label of e , and the rank of v_1 and v_2 . Thus it suffices to prove H preserves the rank of vertices in $Y(\Gamma)$.

Let $\Gamma = \Gamma_1 \circ \Gamma_2 \circ \cdots \circ \Gamma_n$ be the join decomposition of Γ and pick a standard Γ_i -component Y_i in $Y(\Gamma)$. Then Lemma 3.8 (2) implies that $\prod_{i=1}^n p_i : Y(\Gamma) \rightarrow \prod_{i=1}^n Y_i$ is a cubical isomorphism, where $p_i : Y(\Gamma) \rightarrow Y_i$ is the $CAT(0)$ projection. For each i , all Γ_i -components are parallel to each other (see the discussion before Lemma 3.17), and H permutes these Γ_i -components (since H is label-preserving). Thus H respects the product decomposition $Y(\Gamma) \cong \prod_{i=1}^n Y_i$ and induce trivial permutation of the factors. Note that the label of a vertex $y \in Y(\Gamma)$ is the clique spanned by the union of labels of $p_i(y)$ for $1 \leq i \leq n$. Thus it suffices to prove the lemma when $n = 1$. If Γ is discrete, then $Y(\Gamma)$ is a tree. The label-preserving condition implies H preserves rank 0 vertices. If Γ is not discrete, then [44, Theorem 5.24] implies H preserves the rank of all vertices in $Y(\Gamma)$. \square

It follows from the above lemma that the action $H \curvearrowright Y(\Gamma)$ descends to an action $H \curvearrowright |\mathcal{B}|$ through the canonical projection $\pi : Y(\Gamma) \rightarrow |\mathcal{B}|$. Moreover, the action $H \curvearrowright |\mathcal{B}|$ preserves the labelling and rank of vertices.

Lemma 5.2. *There is a finite index subgroup $H' \leq H$ such that $H' \curvearrowright Y(\Gamma)$ is an action without inversion in the sense that if $h \in H'$ fixes a cube in $Y(\Gamma)$, then it fixes the cube pointwise.*

Proof. Pick vertex $u \in \Gamma$. Let \mathcal{H}_u be the collection of hyperplanes in $Y(\Gamma)$ that are labelled by u (see Definition 3.4). Then distinct elements in \mathcal{H}_u have empty intersection. It follows that the dual cube complex to the wall space $(Y(\Gamma), \mathcal{H}_u)$ is a tree, which we denote by T . In other words, T has a vertex for each connected component of $Y(\Gamma) \setminus \mathcal{H}_u$, and an edge when one travels from one component to another by crossing a hyperplane.

Since H is label-preserving, it permutes elements in \mathcal{H}_u . Thus there is an induced action $H \curvearrowright T$. Up to passing to an index 2 subgroup, we assume H acts on T without inversion. Thus $H \curvearrowright Y(\Gamma)$ does not flip any edge in $Y(\Gamma)$ labelled by u . By repeating the above argument for each vertex in Γ , there exists a finite index subgroup $H' \leq H$ which does not flip any edge in $Y(\Gamma)$. Since H' is label-preserving, it acts on $Y(\Gamma)$ without inversion. \square

For comparison, we look at the action of $H \curvearrowright |\mathcal{B}|$. It preserves rank of vertices and label of edges, hence it is already an action without inversion.

From now on, we assume H acts on $Y(\Gamma)$ without inversion. Then the cube complex structure on $Y(\Gamma)$ descends to $K(\Gamma)$, and there is a well-defined labelling of edges of the $Y(\Gamma)/H$ by vertices of Γ . Moreover, each vertex of $K(\Gamma)$ has a well-defined rank and label by Lemma 5.1. Similarly, we can define cube complex structure on $|\mathcal{B}|/H$ as well as labelling of vertices and edges and rank of vertices.

As cube complexes, $K(\Gamma)$ and $|\mathcal{B}|/H$ may not be non-positively curved. For example, let Γ be an edge, then $X(\Gamma) \cong \mathbb{E}^2$. Suppose H acts on \mathbb{E}^2 by translations generated by $(1, 1)$ and $(1, -1)$. Then we can obtain $|\mathcal{B}|/H$ by taking two unit squares and identify them along two consecutive edges.

Lemma 5.3. *Let L be a Γ' -component in $|\mathcal{B}|/H$ and let K be a connected component of $p^{-1}(L)$, where $p : |\mathcal{B}| \rightarrow |\mathcal{B}|/H$. Let $H_K \leq H$ be the stabilizer of K . Then the natural map $i : K/H_K \rightarrow L$ is a cubical isomorphism. In particular, p maps Γ' -components to Γ' -components. Similar statement holds for Γ' -components in $K(\Gamma)$.*

Proof. Note that each connected component of $p^{-1}(L)$ is a Γ' -component. The cube complex structure of K descends to K/H_K , and i is a cubical map which maps cubes in K/H_K to cubes in L of the same dimension. To show i is surjective, it suffices to show $p : p^{-1}(L) \rightarrow L$ is surjective. Since the action of H respects cubical structure, for each point $x \in |\mathcal{B}|/H$ and $y \in p^{-1}(x)$, we can lift each piecewise linear path starting at x to a path starting at y . Thus the surjectivity follows. For the injectivity of i , it suffices to show for any $h \in H$, if $h \cdot K \cap K \neq \emptyset$, then $h \in H_K$. This is true because H is label-preserving. The $K(\Gamma)$ case is similar. \square

We define *standard Γ' -components* in $K(\Gamma)$ (or $|\mathcal{B}|/H$) to be those Γ' -components which contain a rank 0 vertex. Since the canonical projection $\pi : Y(\Gamma) \rightarrow |\mathcal{B}|$ is H -equivariant, it induces a cubical map $\pi_H : K(\Gamma) \rightarrow |\mathcal{B}|/H$.

Remark 5.4. We define a natural map $h_* : H \rightarrow \pi_1(|\mathcal{B}|/H)$ as follows. Pick a base point $\star \in |\mathcal{B}|/H$ and one of its lifts $x \in |\mathcal{B}|$. For each $h \in H$, we pick a path ω_h connecting x and $h \cdot x$, and map h to the element in $\pi_1(|\mathcal{B}|/H, \star)$ represented by the image of ω_h in $|\mathcal{B}|/H$. Since $|\mathcal{B}|$ is simply connected, h_* is well-defined and is a group homomorphism. Moreover, since we can lift piecewise linear paths from $|\mathcal{B}|/H$ to $|\mathcal{B}|$, h_* is surjective.

When H is torsion free, h_* can be defined alternatively as follows. We can pick a base point $\bar{\star} \in K(\Gamma)$ such that $\pi_H(\bar{\star}) = \star$ and identify $\pi_1(K(\Gamma), \bar{\star})$ with H by choosing a lift of $\bar{\star}$ in $Y(\Gamma)$. Then the map $(\pi_H)_* : \pi_1(K(\Gamma), \bar{\star}) \rightarrow \pi_1(|\mathcal{B}|/H, \star)$ coincides with h_* up to conjugation.

Lemma 5.5.

- (1) *The inverse image of a standard Γ' -component in $|\mathcal{B}|/H$ under π_H is a standard Γ' -component.*
- (2) *The π_H -image of a standard Γ' -component is a standard Γ' -component.*

Thus π_H induces a 1-1 correspondence between standard Γ' -components in $K(\Gamma)$ and standard Γ' -components in $|\mathcal{B}|/H$.

Proof. Note that for any edge $e \subset K(\Gamma)$, either $\pi_H(e)$ is a point, or $\pi_H(e)$ is an edge whose label is the same as e . Moreover, π_H induces a bijection between rank 0 vertices (since this is true for the canonical projection $\pi : Y(\Gamma) \rightarrow |\mathcal{B}|/H$). Thus (2) follows from (1). Now we prove (1). Consider the following commuting diagram.

$$\begin{array}{ccc}
Y(\Gamma) & \xrightarrow{\pi} & |\mathcal{B}| \\
\downarrow q & & \downarrow p \\
K(\Gamma) & \xrightarrow{\pi_H} & |\mathcal{B}|/H
\end{array}$$

Pick a standard Γ' -component $L \subset |\mathcal{B}|/H$. Then $p^{-1}(L)$ is a disjoint union of standard Γ' -components. Lemma 5.3 implies H acts transitively on the components of $p^{-1}(L)$. By Lemma 3.14, $\pi^{-1}(p^{-1}(L))$ is also a disjoint union of standard Γ' -components, and H acts transitively on these components. Thus $q \circ \pi^{-1} \circ p^{-1}(L) = \pi_H^{-1}(L)$ is a standard Γ' -component. \square

The following is a consequence of Lemma 3.16.

Lemma 5.6. *For $1 \leq i \leq n$, let L_i be a standard Γ_i -component in $K(\Gamma)$ or $|\mathcal{B}|/H$. If $\cap_{i=1}^n L_i \neq \emptyset$, then it is a disjoint union of standard $\cap_{i=1}^n \Gamma_i$ -components.*

5.2. Trivial holonomy and specialness. For each vertex $v \in \mathcal{P}(\Gamma)$, let P_v be as in Definition 2.5 and let β_{P_v} be as in Definition 3.7. Suppose $H_v \leq H$ is the stabilizer of β_{P_v} and let $\beta_{P_v} = \beta_v \times \beta_v^\perp$ be as in Lemma 3.9. Since H_v preserves this decomposition, there is a factor action $\rho_v : H_v \curvearrowright \beta_v$. The action $H_v \curvearrowright \beta_{P_v}$ is *reducible* if there is a decomposition $H_v = L_v \oplus \mathbb{Z}$ such that L_v acts trivially on the β_v factor and \mathbb{Z} acts trivially on the β_v^\perp factor.

Lemma 5.7. *Let H'_v be a finite index torsion free normal subgroup of H_v and let H_1 and H_2 be two finite index subgroups of H'_v . If the induced action $H_i \curvearrowright \beta_{P_v}$ is reducible for $i = 1, 2$, then*

- (1) *the induced action of $H_1 \cap H_2$ is also reducible;*
- (2) *the induced action of the largest normal subgroup of H_v contained in H_1 is reducible.*

Proof. Since H'_v is torsion free, the action $H'_v \curvearrowright \beta_{P_v}$ is faithful. Then (1) follows readily from the definition. Note that for any $g \in H_v$, $gH_1g^{-1} \subset H'_v$. Moreover, gH_1g^{-1} is reducible. Thus (2) follows from (1). \square

We caution the reader that Lemma 5.7 is not true if we drop the first sentence of the lemma. For example, one can take $H_v = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ acting on \mathbb{R}^2 , where then first and second \mathbb{Z} factor act by translation along the x -axis and y -axis respectively, and the $\mathbb{Z}/2\mathbb{Z}$ acts trivially. Take $H_1 = \langle 1, 0, 0 \rangle \oplus \langle 0, 1, 0 \rangle$ and $H_2 = \langle 1, 0, 1 \rangle \oplus \langle 0, 1, 1 \rangle$. It is easy to check that H_1 and H_2 are reducible, but $H_1 \cap H_2$ is not reducible.

Pick vertex $v \in \Gamma$, it follows from Lemma 3.17 (2) and Lemma 3.15 that each $St(v)$ -component in $Y(\Gamma)$ is of form $\beta_{P_w} = \beta_w \times \beta_w^\perp$ for some vertex $w \in \mathcal{P}(\Gamma)$, hence each $St(v)$ -component $L \subset K(\Gamma)$ is of form β_{P_w}/H_w by Lemma 5.3. If $K(\Gamma)$ is special (recall that we require H to be torsion free in our definition of specialness), then Lemma 4.12 implies that L is a fibre bundle over a branched circle, where the fibres come from the β_w^\perp factor in β_{P_w} . The v -holonomy of L is defined to be the holonomy group of the connection on this fibre bundle induced by the cube complex structure on L . Note that L has trivial v -holonomy if and only if L is isomorphic (as cube complexes) to the product of a v -component in L and $lk(v)$ -component in L , such splitting is called a v -splitting. Since the holonomy group is finite, L has a finite sheet cover with trivial v -holonomy. The v -holonomy of $K(\Gamma)$ is the collection of v -holonomy of all of its $St(v)$ -components.

Definition 5.8. $K(\Gamma)$ has *trivial holonomy* if $K(\Gamma)$ is special and it has trivial v -holonomy for each vertex $v \in \Gamma$. Similarly, if $S'(\Gamma)$ is a finite cover of the Salvetti complex $S(\Gamma)$, we define the v -holonomy of its $St(v)$ -component in a similar way. Again, if all such holonomy vanishes, then $S'(\Gamma)$ has *trivial holonomy*.

Remark 5.9. We caution the reader that the notion of trivial holonomy defined here is different from Haglund's notion [35, Defition 5.6] in the studying of right-angled building. In particular, our notion is not stable when passing to subgroups while Haglund's notion is.

Suppose $K(\Gamma)$ is special. Then $K(\Gamma)$ has trivial holonomy if and only if $H_w \curvearrowright \beta_{P_w}$ is reducible for each vertex $w \in \mathcal{P}(\Gamma)$. This together with Lemma 5.7 imply the following result.

Lemma 5.10. *Let H' be a finite index torsion free normal subgroup of H . Suppose there exist finite index subgroups $H_1, H_2 \leq H'$ such that $Y(\Gamma)/H_i$ has trivial holonomy for $i = 1, 2$. Then $Y(\Gamma)/(H_1 \cap H_2)$ has trivial holonomy. In particular, let $N \triangleleft H$ be the largest normal subgroup contained in H_1 , then $Y(\Gamma)/N$ has trivial holonomy.*

Lemma 5.11. *Let L be a Γ' -component in $|\mathcal{B}|$. Suppose Γ' admit a join decomposition $\Gamma' = \Gamma_1 \circ \Gamma_2$ where Γ_1 is a clique with its vertices denoted by $\{v_i\}_{i=1}^n$. Let $L = \prod_{i=1}^n L_i \times L_{n+1}$ be the product decomposition such that L_i corresponds to v_i for $1 \leq i \leq n$ and L_{n+1} corresponds to Γ_2 . Let $H_L \leq H$ be the stabilizer of L . If $K(\Gamma)$ has trivial holonomy, then $H_L = \oplus_{i=1}^{n+1} H_i$ such that H_i acts trivially on L_j if $i \neq j$, and $H_i \cong \mathbb{Z}$ for $1 \leq i \leq n$.*

Proof. First we look at the case $\Gamma_1 = \{v\}$ and $\Gamma_2 = lk(v)$. Then L is a standard component by Lemma 3.17 (2) and $\pi^{-1}(L) = \beta_{P_w}$ for some vertex $w \in \mathcal{P}(\Gamma)$ by Lemma 3.14 and Lemma 3.15 (π is the canonical projection). Moreover, the stabilizer is exactly H_L . Since $K(\Gamma)$ has trivial holonomy, the action $H_L \curvearrowright \beta_{P_w}$ is reducible. Since π respects the product decomposition of β_{P_w} , the action $H_L \curvearrowright L$ has the required decomposition. Now we look at the case $\Gamma_1 = \{v\}$ and Γ_2 is any induced subgraph of $lk(v)$. This follows from the previous case by consider the $St(v)$ -component that contains L . In general, for each vertex $v_i \in \Gamma_1$, Γ' can be written as a join of $\{v_i\}$ and a induced subgraph of $lk(v_i)$. Thus we can induct on the number of vertices in Γ_1 and apply case 2 to reduce the number of vertices. \square

Lemma 5.12. *If $K(\Gamma)$ has trivial holonomy, then $|\mathcal{B}|/H$ is a compact special cube complex.*

Proof. First we show $|\mathcal{B}|/H$ is a non-positively curved cube complex. Let $x \in |\mathcal{B}|$ be a vertex of rank n and let $p : |\mathcal{B}| \rightarrow |\mathcal{B}|/H$ be the projection map. It suffices to show the link of $p(x)$ in $|\mathcal{B}|/H$ is flag.

Suppose x is labelled by Δ_x with its vertex set denoted by $\{v_i\}_{i=1}^n$. Let $\Delta_x^\perp \subset \Gamma$ be the induced subgraph spanned by vertices in Γ which are adjacent to each vertex in Δ_x , and let L be the $\Delta_x \circ \Delta_x^\perp$ -component containing x . For each $1 \leq i \leq n$, let L_i be the v_i -component that contains x . Then each L_i is a infinite tree of diameter 2. Let L_{n+1} be the Δ_x^\perp -component that contains x . Then L admits a product decomposition $L = \prod_{i=1}^n L_i \times L_{n+1}$. By the construction of $|\mathcal{B}|$, each edge in $|\mathcal{B}|$ which contains x is labelled by a vertex in $\Delta_x \circ \Delta_x^\perp$, thus a small neighbourhood of x in $|\mathcal{B}|$ is contained in L . Let $H_L \leq H$ be the stabilizer of L . By Lemma 5.3, it suffices to show the link of $p(x)$ in L/H_L is flag.

By Lemma 5.11, $H_L = \oplus_{i=1}^{n+1} H_i$ such that H_i acts trivially on L_j if $i \neq j$, and $H_i \cong \mathbb{Z}$ for $1 \leq i \leq n$. Then H/H_L and $\prod_{i=1}^{n+1} L_i/H_i$ are isomorphic as cube complexes. We represent x as $(x_1, x_2, \dots, x_{n+1})$ with respect to the product decomposition of L . Note that different edges in H_{n+1} that contains x_{n+1} have different labels (actually they are in 1-1 correspondence with vertices in Δ_x^\perp). Since the action $H_{n+1} \curvearrowright L_{n+1}$ is label-preserving, any element in H_{n+1} which fixes x_{n+1} also fixes each edge emanating from x , hence fixes a small neighbourhood around x_{n+1} . Thus the link of $p(x_{n+1})$ in L_{n+1}/H_{n+1} is isomorphic to the link of x_{n+1} in L_{n+1} , hence is flag. On the other hand, all L_i/H_i 's are trees for $1 \leq i \leq n$. Thus the link of $p(x)$ in L/H_L is flag.

Now we show $|\mathcal{B}|/H$ is special. We can orient each edge of $|\mathcal{B}|$ from vertex of lower rank to vertex of higher rank. This orientation is compatible with parallelism. Since H preserves the rank of vertices, it preserves this orientation, hence hyperplanes in $|\mathcal{B}|/H$ are two-sided. They do not self-intersect since the action $H \curvearrowright |\mathcal{B}|$ preserve the labelling of hyperplanes. Inter-osculation is ruled out by Lemma 3.17 (3).

It remains to show hyperplanes in $|\mathcal{B}|/H$ do not self-osculte. Pick a hyperplane $h \subset |\mathcal{B}|/H$ and let v be the label of an edge dual to h (all edges dual to h have the same label). Then Lemma 3.17 (3) implies there is a $St(v)$ -component $L' \subset |\mathcal{B}|/H$ such that L' contains each cube which intersects h . It suffices to show h does not self-osculte in L' . Lemma 5.11 implies L' admits a product decomposition $L' \cong L'_1 \times L'_2$ which is induced by the join decomposition $St(v) = \{v\} \circ lk(v)$. Note that L'_1 is a finite tree of diameter 2 and h is dual to some edge in the L'_1 factor, thus h does not self-osculte. \square

Definition 5.13. For each special $K(\Gamma)$ we associate a cube complex, which is called the *reduction* of $K(\Gamma)$ and is denoted by $S_K(\Gamma)$, as follows. Let $f : X(\Gamma)^{(0)} \rightarrow Y(\Gamma)$ be the bijection between $X(\Gamma)^{(0)}$ and rank 0 vertices of $Y(\Gamma)$ in Definition 3.2. Recall that we have identified $X(\Gamma)^{(0)}$ with $G(\Gamma)$, thus f induces an action $H \curvearrowright G(\Gamma)$ by flat-preserving bijections. Lemma 4.12 and Theorem 4.4 imply that up to pre-composing f with a flat-preserving bijection, we can assume $H \curvearrowright G(\Gamma)$ is an action by left translations. Using the identification $X(\Gamma)^{(0)} \cong G(\Gamma)$ again, we have an isometric action $H \curvearrowright X(\Gamma)$. Define $S_K(\Gamma) = X(\Gamma)/H$. Note that $S_K(\Gamma)$ is a finite sheet cover of the Salvetti complex $S(\Gamma)$.

Pick vertex $w \in \mathcal{P}(\Gamma)$. Note that $f : P_w^{(0)} \cong \mathbb{Z}_w \times Q_v^{(0)} \rightarrow \beta_{P_w} \cong \beta_w \times \beta_w^\perp$ respects the product decompositions. Thus $H_w \curvearrowright \beta_{P_w}$ is reducible if and only if $H_w \curvearrowright P_w^{(0)}$ is reducible. Hence $K(\Gamma)$ has trivial holonomy if and only if its reduction $S_K(\Gamma)$ has trivial holonomy.

Let \mathcal{C}_K be the category whose objects are standard Γ' -components in finite covers of $K(\Gamma)$ (Γ' can be any induced subgraph of Γ), and morphisms are label and rank preserving local isometries. Let \mathcal{C}_S be the category whose objects are Γ' -components in finite covers of the reduction $S_K(\Gamma)$, and morphisms are label-preserving local isometries. Note that f is H -equivariant, and it maps the 0-skeleton of a Γ' -component in $X(\Gamma)$ to vertices of rank 0 in a standard Γ' -component of $Y(\Gamma)$. Thus the following holds.

Lemma 5.14. *The map f induces a functor Φ_f from \mathcal{C}_S to \mathcal{C}_K . Moreover, an object in \mathcal{C}_S has trivial holonomy if and only if its image in \mathcal{C}_K has trivial holonomy.*

5.3. Coverings of complexes with trivial holonomy.

Lemma 5.15. *Suppose $K(\Gamma)$ has trivial holonomy. Let $L \subset K(\Gamma)$ be a Γ' -component and let $L' \rightarrow L$ be a finite cover with trivial holonomy. Then the canonical complete $\mathbb{C}(L', K(\Gamma))$ also has trivial holonomy. Moreover, L' is wall-injective in $\mathbb{C}(L', K(\Gamma))$.*

Proof. Let Λ be the intersection graph of $K(\Gamma)$. Recall that $\mathbb{C}(L', K(\Gamma))$ is the pull-back defined as follows (recall that we require $K(\Gamma)$ to be special in our definition of trivial holonomy).

$$\begin{array}{ccc} \mathbb{C}(L', K(\Gamma)) & \longrightarrow & \mathbb{C}(L', S(\Lambda)) \\ c \downarrow & & b \downarrow \\ K(\Gamma) & \xrightarrow{a} & S(\Lambda) \end{array}$$

Pick $St(v)$ -component $N \subset K(\Gamma)$. Since $K(\Gamma)$ has trivial holonomy, let $N = N_1 \times N_2$ be the v -splitting of N , where N_1 is a $\{v\}$ -component in N . Note that $a(N) = a(N_1) \times a(N_2)$ and $a|_N$ splits as a product of two maps. Let $N' \subset \mathbb{C}(L', S(\Lambda))$ be a lift of $a(N) \subset S(\Lambda)$. We claim N' has a similar cubical splitting as $a(N)$ and $b|_{N'}$ splits as a product of two maps. Then one deduce from this claim and the definition of pull-back that each component of $c^{-1}(N)$ admits a similar cubical splitting, which implies $\mathbb{C}(L', K(\Gamma))$ has trivial holonomy.

Now we prove the claim. It follows from the definition of $\mathbb{C}(L', S(\Lambda))$ that N' is of form $\mathbb{C}(N' \cap L', a(N))$, here the map $N' \cap L' \rightarrow a(N)$, which we denoted by i , is the restriction of the map $L' \rightarrow L \xrightarrow{a} S(\Lambda)$ to $N' \cap L'$. By Lemma 2.11, it suffices to show i has a splitting which is compatible with $a(N) = a(N_1) \times a(N_2)$. Let $p : L' \rightarrow L$ be the covering map. Note that $p(N' \cap L')$ is a connected component in the wall projection $W\text{Proj}_{K(\Gamma)}(N \rightarrow L)$. If no edge of $p(N' \cap L')$ is parallel to some edge in N_1 , then the image of i is inside an $a(N_2)$ -slice of $a(N)$ and i splits trivially. If there is an edge $e \subset p(N' \cap L')$ parallel to an edge in N_1 , then $e \subset N$. We also deduce that $p(N' \cap L') \subset N$. Thus $N' \cap L'$ is contained in some $St(v, \Gamma')$ -component of L' . Since L' has trivial holonomy, this component admits a v -splitting. Since $N' \cap L'$ is locally convex, it inherits a splitting, and i also splits as required.

Since each $St(v)$ -component of $K(\Gamma)$ has a v -splitting, the assumption of Lemma 2.18 is satisfied. Thus L' is wall-injective in $\mathbb{C}(L', K(\Gamma))$. \square

Lemma 5.16. *Suppose $K(\Gamma)$ is special. Then $K(\Gamma)$ has a finite cover which has trivial holonomy.*

Proof. By Lemma 5.14, we can assume $K(\Gamma)$ is a finite cover of $S(\Gamma)$. We induct on the number of vertices in Γ . The case where Γ has only one vertex is trivial. If there exists vertex $v \in \Gamma$ such that $\Gamma = St(v)$, then up to passing to a finite sheet cover we can assume $K(\Gamma)$ is isomorphic to a product of a circle and a $lk(v)$ -component. However, by induction, the $lk(v)$ -component has a finite cover with trivial holonomy, thus $K(\Gamma)$ has the required cover. Now we assume $St(v) \subsetneq \Gamma$ for any vertex $v \in \Gamma$.

By induction, there exists a finite sheet cover A_v of the Salvetti complex $S(St(v))$ such that A_v has trivial holonomy and A_v factors through any $St(v)$ -components of $K(\Gamma)$. By Lemma 5.10, we can assume A_v is a regular cover. Let K_v be the pull back as follows.

$$\begin{array}{ccc}
K_v & \longrightarrow & C(A_v, S(\Gamma)) \\
\downarrow & & \downarrow \\
K(\Gamma) & \longrightarrow & S(\Gamma)
\end{array}$$

Then each $St(v)$ -component of K_v has trivial v -holonomy.

We claim if we already know $K(\Gamma)$ has trivial v' -holonomy for a vertex $v' \neq v$, then K_v also has trivial v' -holonomy. Then the lemma follows by repeating the above argument for each vertex in Γ . However, this claim follows from $C(A_v, S(\Gamma))$ has trivial holonomy (the proof is identical to Lemma 5.15) and Lemma 5.7. \square

6. CONTROL OF RETRACTION IMAGE

6.1. Wall projections in branched complexes. Let $K(\Gamma)$, $Y(\Gamma)$, H and $|\mathcal{B}|$ be as in Section 5. In this section we study wall projections in $K(\Gamma)$. Assuming hyperbolicity, a powerful tool for this purpose is the following connected intersection theorem proved by Haglund and Wise.

Theorem 6.1. ([39, Theorem 4.23]) *Let X be a compact special cube complex whose universal cover is Gromov-hyperbolic. Let (B_0, \dots, B_n, A) be connected locally convex subcomplexes containing the basepoint of X . Suppose that $A \subset \cap_{j=0}^n B_j$.*

Then there is a based finite cover \bar{X} and base elevations $\bar{B}_0, \dots, \bar{B}_n$ with $\bar{A} \cong A$, such that $\cap_{j \in J} \bar{B}_j$ is connected for each $J \subset \{0, \dots, n\}$.

However, this result does not apply to $K(\Gamma)$ directly since $Y(\Gamma)$ is not hyperbolic in general. We first deal with this issue.

For a $CAT(0)$ cube complex X , we can endow each cube with the l^1 -metric, which gives rise to a l^1 -metric on X . An l^1 -flat in X is an isometric embedding $\mathbb{E}^n \rightarrow X$ with respect to the l^1 -metric such that its image is a subcomplex of X . An l^1 -geodesic is a 1-dimension l^1 -flat. The following is a version of Eberlein's theorem (see [15, Theorem II.9.33]) in the cubical setting.

Lemma 6.2. *Let X be a proper and cocompact $CAT(0)$ cube complex. Then X is Gromov-hyperbolic if and only if X does not contain a 2 dimensional l^1 -flat.*

Proof. It suffices to show the only if direction. Suppose X is not Gromov-hyperbolic, then [49, Theorem C] implies X contains a 2-flat F (with respect to the $CAT(0)$ metric). Let L be the smallest convex subcomplex of X that contains F . Then [43, Corollary 3.5] implies L admits a splitting $L = K_1 \times K_2 \times \dots \times K_n \times K$ such that $n \geq 2$ and each K_i contains a geodesic line. Since each K_i is uniformly locally finite, it is not hard to approximate a geodesic line in K_i by an l^1 -geodesic line. Hence X contains a 2-dimension l^1 -flat, which is a contradiction. \square

Lemma 6.3. *Suppose $K(\Gamma)$ has trivial holonomy. If Γ does not contain any induced 4-cycle, then the universal cover X of $|\mathcal{B}|/H$ does not contain a 2 dimensional l^1 -flat, hence hyperbolic.*

Proof. X has an induced edge-labelling from $|\mathcal{B}|/H$. Let $\ell \subset X$ be a l^1 -geodesic line and let $\Gamma_\ell \subset \Gamma$ be the set of labels of edges of ℓ . We claim the diameter of Γ_ℓ is ≥ 2 . Suppose the contrary is true. Then there exists a clique $\Delta \subset \Gamma$ which contains Γ_ℓ . It follows from Lemma 5.3 and Lemma 5.11 that each Δ -component in $|\mathcal{B}|/H$ is a product of trees of diameter 2, hence contractible. Thus each Δ -component in X is bounded, which is contradictory to that it contains an l^1 -geodesic line ℓ .

Since $K(\Gamma)$ has trivial holonomy, $|\mathcal{B}|/H$ is non-positively curved by Lemma 5.12. Thus by Lemma 6.2, it suffices to show X can not contain any 2 dimensional l^1 -flat. Suppose the contrary is true and let ℓ_1 and ℓ_2 be two l^1 -geodesic lines which spans a 2-flat. Note that Lemma 3.17 (3) is true for $|\mathcal{B}|/H$, hence for X . Thus each vertex in Γ_{ℓ_1} is adjacent to every vertex in Γ_{ℓ_2} . By the previous claim, we can find a pair of non-adjacent vertices in both Γ_{ℓ_1} and Γ_{ℓ_2} , which gives rise to an induced 4-cycle in Γ . \square

Corollary 6.4. *Suppose $K(\Gamma)$ has trivial holonomy and Γ does not contain induced 4-cycle. Let (B_0, \dots, B_n, A) be a collection of standard components containing the basepoint of $K(\Gamma)$. Suppose that $A \subset \cap_{j=0}^n B_j$.*

Then there is a based finite cover $\bar{K}(\Gamma)$ and base elevations $\bar{B}_0, \dots, \bar{B}_n$ with $\bar{A} \cong A$, such that $\cap_{j \in J} \bar{B}_j$ is connected for each $J \subset \{0, \dots, n\}$.

Proof. Let $\pi_H : K(\Gamma) \rightarrow |\mathcal{B}|/H$ be the map induced by the canonical projection. Suppose $\pi_H(B_i) = D_i$ and $\pi_H(A) = C$. Then (D_0, \dots, D_n, C) are standard components in $|\mathcal{B}|/H$ by Lemma 5.5, in particular they are locally convex by Lemma 2.3. It follow from Lemma 5.12 and Lemma 6.3 that $|\mathcal{B}|/H$ satisfies the assumption of Theorem 6.1. Thus there is a based finite cover $\bar{L} \rightarrow |\mathcal{B}|/H$ and based elevations $\bar{D}_0, \dots, \bar{D}_n$ with $\bar{C} \cong C$, such that $\cap_{j \in J} \bar{D}_j$ is connected for each $J \subset \{0, \dots, n\}$. Let $\bar{K}(\Gamma)$ be the based cover of $K(\Gamma)$ corresponding to the subgroup $H' = (\pi_H)_*^{-1}(\pi_1(\bar{L}, \star))$. This gives rise to the following commuting diagram between based spaces.

$$\begin{array}{ccc} (\bar{K}(\Gamma), \star) & \xrightarrow{\pi'} & (\bar{L}, \star) \\ \downarrow & & \downarrow \\ (K(\Gamma), \star) & \xrightarrow{\pi_H} & (|\mathcal{B}|/H, \star) \end{array}$$

Let $(\bar{B}_0, \dots, \bar{B}_n, \bar{A})$ be the based lifts in $\bar{K}(\Gamma)$. Then they are also standard components, $\pi'(\bar{B}_i) = \bar{D}_i$ and $\pi'(\bar{A}) = \bar{C}$. Thus $\bar{A} \cong A$. Recall that π_H induces a surjection on the fundamental groups, then π' is exactly the H' -equivariant quotient of the canonical projection π (see Remark 5.4). Then the connectedness of $\cap_{j \in J} \bar{B}_j$ follows from Lemma 5.6 and Lemma 5.5. \square

Corollary 6.5. *Let Γ' be a graph without induced 4-cycles. Pick vertex $v \in \Gamma'$ and let $\Gamma \subset \Gamma'$ be the induced subgraph spanned by vertices in $\Gamma' \setminus \{v\}$. Let $\Gamma_0 = lk(v, \Gamma') \subset \Gamma$. Suppose $K = K(\Gamma)$ is special and pick a base point $p \in K$. Let $i : A \rightarrow K$ be a local isometry such that*

- (1) *the image of A is a standard Γ_0 -component which contains the base point p ;*
- (2) *the map $i : A \rightarrow i(A)$ is a covering map of finite degree.*

Then there exists a finite cover $A_0 \rightarrow A$ such that for any further finite cover $\bar{A} \rightarrow A_0$ with trivial holonomy, there exists a finite directly special cover $\bar{K} \rightarrow K$ and an embedding $\bar{A} \rightarrow \bar{K}$ such that they fit into the following commutative diagram:

$$\begin{array}{ccc} \bar{A} & \longrightarrow & \bar{K} \\ \downarrow & & \downarrow \\ A & \longrightarrow & K \end{array}$$

Moreover, the following statements are true.

- (1) All elevations of $A \rightarrow K$ to \bar{K} are embedded.
- (2) \bar{A} is wall-injective in \bar{K} .
- (3) Let $\bar{B} \subset \bar{K}$ be any Γ_0 -component distinct from \bar{A} . Then the wall projection $\text{WProj}_{\bar{K}}(\bar{B} \rightarrow \bar{A})$ is a disjoint union of branched tori of various dimension and isolated points.

In the following proof, for any vertex $v \in \Gamma$, we always consider its link $lk(v)$ and its closed star $St(v)$ inside Γ rather than Γ' .

Proof. By Lemma 2.10, we assume K is directly special. Let K_1 be the canonical completion $C(A, K)$. Take $K_2 \rightarrow K$ to be the smallest regular cover factoring through each component of K_1 (more precisely, each component of K_1 gives rise to a finite index subgroup of $\pi_1(K, p)$, and K_2 is the regular cover corresponding to the intersection of all these subgroups, as well as their conjugates). It is clear that all elevations of $A \rightarrow K$ to K_2 are injective. Let $K_3 \rightarrow K_2$ be a finite cover which has trivial holonomy (Lemma 5.16). We can assume $K_3 \rightarrow K$ is regular by Lemma 5.10. Let $A_0 \rightarrow K_3$ be an elevation of $A \rightarrow K$. We claim A_0 satisfies our condition.

Let $C(\bar{A}, K_3)$ be the canonical completion with respect to the map $\bar{A} \rightarrow A_0 \rightarrow K_3$ and let \hat{K} be its main component. Then \hat{K} has trivial holonomy and \bar{A} is wall injective in \hat{K} by Lemma 5.15.

Suppose $\{v_i\}_{i=1}^n$ are the vertices in Γ_0 . For each i , let $\Gamma_i \subset \Gamma$ be the induced subgraph spanned by $St(v_i) \cup \Gamma_0$. We claim $\Gamma_i \cap \Gamma_j = \Gamma_0$ if v_i and v_j are not adjacent. If there exists a vertex $u \in (\Gamma_i \cap \Gamma_j) \setminus \Gamma_0$, then u, v_i, v_j and v will form a induced 4-cycle in Γ' , which yields a contradiction.

By Corollary 6.4, there exists a finite cover $\bar{K} \rightarrow \hat{K}$ such that we can find an elevation of \bar{A} in \bar{K} which is isomorphic to \bar{A} (hence we also denote this elevation by \bar{A}) such that the standard Γ_i -component $\bar{K}_i \subset \bar{K}$ that contains \bar{A} satisfies $\bigcap_{i \in I} \bar{K}_i$ is connected for any $I \subset \{1, \dots, n\}$. Thus by Lemma 5.6 and the previous paragraph, $\bar{K}_i \cap \bar{K}_j = \bar{A}$ if v_i and v_j are not adjacent. We claim \bar{A} and \bar{K} satisfy the requirements of the corollary.

Since the covering map $\bar{K} \rightarrow K$ factors through K_2 , every elevation of A to \bar{K} is embedded. Since \bar{K} covers K , it is directly special. Since \bar{A} is wall-injective in \hat{K} (Lemma 2.18), it is wall-injective in \bar{K} . Let $\bar{B} \neq \bar{A}$ be another Γ_0 -component. Pick two distinct edges e_i and e_j in $\text{WProj}_{\bar{K}}(\bar{B} \rightarrow \bar{A})$ and suppose they are labelled by v_i and v_j in Γ_0 respectively. Then $\bar{B} \subset \bar{K}_i$ and $\bar{B} \subset \bar{K}_j$. Thus $v_i = v_j$ or they are adjacent, otherwise we would have $\bar{K}_i \cap \bar{K}_j = \bar{A}$, which contradicts $\bar{B} \neq \bar{A}$. Thus the labels of all edges in $\text{WProj}_{\bar{K}}(\bar{B} \rightarrow \bar{A})$ are contained in a clique of Γ . On the other hand, by the definition of wall projection, if each edge in a corner of some cube is contained $\text{WProj}_{\bar{K}}(\bar{B} \rightarrow \bar{A})$, then this cube and the smallest branched torus containing this cube is contained in $\text{WProj}_{\bar{K}}(\bar{B} \rightarrow \bar{A})$. Thus (3) follows. \square

6.2. The modified completions and retractions. Let $K = K(\Gamma) = Y(\Gamma)/H$ be a branched complex which is directly special. Recall that parallel edges have the same label, thus there is a well-defined labelling for hyperplanes of K . Note that each edge $e \subset K$ is contained in a v -component ($v \in \Gamma$ is the label of e), which is a branched circle. If e is contained in the core of this branched circle, then e is called a *core edge*. Since a core edge is parallel to another core edge, it makes sense to talk about *core hyperplanes* in K . We orient each edge in K such that

- (1) the orientation respects parallelism between edges;
- (2) for any circle in K made of core edges, the orientation of each edge in the circle fits together to give an orientation of the circle.

Condition (2) is possible by Lemma 4.12. Namely for each standard branched line in $Y(\Gamma)$, we can choose an orientation for its core. And we can require this choice is compatible with parallelism and is H -equivariant, thus it descends to orientation of corresponding circles in K .

Two hyperplanes of K are *equivalent* if and only if they are both core hyperplanes and they are dual to the same v -component for some vertex $v \in \Gamma$. We claim this is indeed an equivalent relationship. Suppose for $i = 1, 2$, h and h_i are core hyperplanes which are dual to the same v_i -component. Then $v_1 = v_2$, and h and h_i are in the same $St(v_i)$ -component. Thus h, h_1 and h_2 are in the same $St(v_1)$ -component. Recall that each $St(v_1)$ -component is of form $\beta_{P_{\bar{v}}}/H_{\bar{v}}$ for some vertex $\bar{v} \in \mathcal{P}(\Gamma)$ (Lemma 3.17 (2), Lemma 3.15 and Lemma 5.3). Then h_1 and h_2 are equivalent by Lemma 3.9.

For two equivalent classes of hyperplanes \mathcal{C}_1 and \mathcal{C}_2 , if an element in \mathcal{C}_1 crosses an element in \mathcal{C}_2 , then each element in \mathcal{C}_1 will cross every element in \mathcal{C}_2 . Let Γ_K be a graph on the equivalence classes of hyperplanes in K such that vertices are adjacent if and only if the corresponding equivalent classes cross.

For each class of hyperplanes \mathcal{C} , we define whether \mathcal{C} (*directly or indirectly*) *self-osculates* in the same way as the beginning of Section 2.2, except we change the definition of g_v there to be the graph made of all edges which are dual to some hyperplane in \mathcal{C} and contain v . Similarly, the notion of *inter-osculation* is also well-defined for two classes of hyperplanes. It follows from our definition of the equivalence relationship, as well as our choice of the edge orientation of K , that each equivalent class of hyperplanes does not directly self-osculate, and no two classes inter-osculate (note that Lemma 3.17 (3) is also true for $K(\Gamma)$, and this excludes the inter-osculation of two classes).

There is a natural map $K \rightarrow S(\Gamma_K)$ by sending an oriented edge \vec{a} in K to the oriented edge in $S(\Gamma_K)$ that corresponds to the hyperplane dual to \vec{a} . The above discussion implies that this map is a local isometry. Suppose $A \subset K$ is an Γ' -component for $\Gamma' \subset \Gamma$, and let $A \rightarrow K \rightarrow S(\Gamma_K)$ be the composition map. Let $C'(A, K)$ be the pullback which fits into the following diagram:

$$\begin{array}{ccc} C'(A, K) & \longrightarrow & C(A, S(\Gamma_K)) \\ \downarrow & & \downarrow \\ K & \longrightarrow & S(\Gamma_K) \end{array}$$

Here $C(A, S(\Gamma_K))$ is the canonical completion defined in Section 2.3. An edge of $S(\Gamma_K)$ is a *core edge* if it arises from a family of core hyperplanes. Inverse image of core edges in $S(\Gamma_K)$ are defined to be *core edges* in $C(A, S(\Gamma_K))$.

There is a natural copy of A inside $C'(A, K)$. Let $r : C(A, S(\Gamma_K)) \rightarrow A$ be the canonical retraction. Note that if $e \subset S(\Gamma_K)$ is a core edge, then its inverse image under the map $A \rightarrow S(\Gamma_K)$ is a disjoint union of circles or isolated points; if e is not a core edge, then its inverse image is a disjoint union of edges or points (since $K(\Gamma)$ is directly special). It follows that r is a cubical map. Moreover,

Lemma 6.6. *Let $e \subset C(A, S(\Gamma_K))$ be a core edge. If $r(e)$ is also an edge, then $r(e) = e$.*

Let r' be the composition $C'(A, K) \rightarrow C(A, S(\Gamma_K)) \rightarrow A$ which is also a retraction. Note that $C'(A, K)$ is different from $C(A, K)$ in general. $C'(A, K)$ is called the *modified completion* and r' is called the *modified retraction*.

Lemma 6.7. *Let $D \subset K$ be a Λ -component for a induced subgraph $\Lambda \subset \Gamma$ and let \hat{D} denote the preimage of D in $C'(A, K)$. Then $r'(\hat{D}) \subset \text{WProj}_K(D \rightarrow A)$.*

Proof. By the definition of pull-back, edges of $C'(A, K)$ are of form (b_1, b_2) where $b_1 \subset K$ and $b_2 \subset C(A, S(\Gamma_K))$ are sent to the same edge of $S(\Gamma_K)$. Moreover, $r'(b_1, b_2) = r(b_2)$ where $r : C(A, S(\Gamma_K)) \rightarrow A$ is the canonical retraction. Suppose $b_1 \subset D$ and $r(b_2) = b'_2$ is an edge.

Case 1: b_2 is not a core edge. Then b_1 is not a core edge. In this case, b_2 and b'_2 are mapped to the same edge in $S(\Gamma_K)$. Hence the same is true for b'_2 and b_1 . Hence they are parallel.

Case 2: b_2 is a core edge. Then b_1 is also a core edge. Moreover $b_2 = b'_2 \subset A$ by Lemma 6.6. Thus the hyperplane dual to b_1 and the hyperplane dual to b_2 are in the same equivalent class. For $i = 1, 2$, let C_i be the circle made of core edges that contains b_i . Then $\text{WProj}_K(C_1 \rightarrow C_2) = C_2$. Moreover, $C_2 \subset A$ and $C_1 \subset D$. Thus $b'_2 \subset \text{WProj}_K(D \rightarrow A)$. \square

Remark 6.8. It follows from the above proof that if $r'(e)$ is a core edge in A , then e has to be a core edge in \hat{D} .

Lemma 6.9. *Let $C \subset K$ be a circle made of core edges. Then the inverse image of C under $C'(A, K) \rightarrow K$ is a disjoint union of circles whose combinatorial length are equal to the length of C .*

Proof. Suppose $\text{length}(C) = l$. Let e be the image of C under $K \rightarrow S(\Gamma_K)$ (e is an edge). We claim the inverse image of e under $C(A, S(\Gamma_K)) \rightarrow S(\Gamma_K)$ consists of circles of length 1 or l . It suffices to show the inverse image of e under $A \rightarrow S(\Gamma_K)$ consists of circles of length l or isolated points. Suppose $C' \subset A$ is a circle mapped to e . Then hyperplanes dual to edges in C' are equivalent to hyperplanes dual to edges in C , hence edges of C and C' are labelled by the same vertex $v \in \Gamma$, moreover, C and C' are in the same $St(v)$ -component. Thus they have the same length. Now the lemma follows from the claim and the construction of $C'(A, K)$. \square

One can compare the above result with Remark 2.17.

Corollary 6.10. *Let $C \subset C'(A, K)$ be a circle made of core edges and let $r' : C'(A, K) \rightarrow A$ be the modified retraction. Then either $r'(C)$ is a point, or $r'|_C$ is an isomorphism. More precisely, let $C_1 \subset K$ be a circle made of core edges.*

- (1) *If $\text{WProj}_K(C_1 \rightarrow A)$ does not contain any edge, then r' sends each component in the inverse image of C_1 under $C'(A, K) \rightarrow K$ to a point.*
- (2) *If $\text{WProj}_K(C_1 \rightarrow A)$ contains at least one edge, then either r' maps a component as above to a point, or r' maps it isometrically into A . And there exists at least one component such that the latter case happens.*

Proof. Suppose edges in C_1 are labelled by v . Then (1) follows by applying Lemma 6.7 to the v -component that contains C_1 . Under the assumption in (2), there exists a circle made of core edges $C_2 \subset A$ such that $\text{WProj}_K(C_1 \rightarrow C_2) = C_2$. The images of C_1 and C_2 under $K \rightarrow S(\Gamma_K)$ give rise to the same edge $e \subset S(\Gamma_K)$. Thus the pair (C_1, C_2) represents a circle in $C'(A, K)$ which is mapped isometrically to C_2 under the map r' . The rest of (2) follows from the proof of Lemma 6.9. \square

Let A be as above. Now we want to temporarily forget about the fact that A is inside a larger space K . We define two hyperplanes of A are *equivalent* if and only if there exists a vertex $v \in \Gamma'$ such that these two hyperplanes are dual to core edges the same v -component. Let Γ_A be a graph on the equivalence classes of hyperplanes in A defined in a similar way to Γ_K . We define $C'(A, A)$ to be the pullback that fits into the following diagram:

$$\begin{array}{ccc} C'(A, A) & \longrightarrow & C(A, S(\Gamma_A)) \\ \downarrow & & \downarrow \\ A & \longrightarrow & S(\Gamma_A) \end{array}$$

Lemma 6.11. *If the inclusion $A \rightarrow K$ induces an injective map of equivalence classes of hyperplanes, then there is a canonical isomorphism ϕ from $C'(A, A)$ to the inverse image of A under the covering map $C'(A, K) \rightarrow K$. Moreover, let r'_1, r'_2 be modified retractions, then the following diagrams commutes:*

$$\begin{array}{ccc} C'(A, A) & \xrightarrow{\phi} & C'(A, K) \\ \downarrow & & \downarrow \\ A & \longrightarrow & K \end{array} \quad \begin{array}{ccc} C'(A, A) & \xrightarrow{\phi} & C'(A, K) \\ & \searrow r'_1 & \downarrow r'_2 \\ & & A \end{array}$$

Proof. The proof is similar to [39, Lemma 3.13]. By assumption, there is a natural embedding $S(\Gamma_A) \rightarrow S(\Gamma_K)$ (this may not be a local-isometry). This induces an embedding $C(A, S(\Gamma_A)) \rightarrow C(A, S(\Gamma_K))$ such that the following diagram commutes (r, r' are canonical retractions):

$$\begin{array}{ccc} C(A, S(\Gamma_A)) & \rightarrow & C(A, S(\Gamma_K)) \\ & \searrow r' & \downarrow r \\ & & A \end{array}$$

Moreover, $C(A, S(\Gamma_A))$ is the inverse image of $S(\Gamma_A)$ under $C(A, S(\Gamma_K)) \rightarrow S(\Gamma_K)$. Now the lemma follows from the definition of pullback and modified retraction. \square

Lemma 6.12. *The assumption of Lemma 6.11 is satisfied if A is wall-injective in X .*

Proof. Suppose there are two different classes of core hyperplanes \mathcal{C}_1 and \mathcal{C}_2 which are mapped to the same class \mathcal{C} of hyperplanes of X . Then there is a circle made of core edges $C \subset A$ such that \mathcal{C}_1 and \mathcal{C} are made of hyperplanes dual to edges in C . Since A is wall-injective, there is 1-1 correspondence between elements in \mathcal{C}_1 and elements in \mathcal{C} . Thus there exist a hyperplane in \mathcal{C}_1 and a hyperplane in \mathcal{C}_2 which are mapped to the same hyperplane in \mathcal{C} . This yields a contradiction. \square

Remark 6.13. Let $A \rightarrow A'$ be a label-preserving isomorphism. Then it induces an isomorphism $S(\Gamma_A) \rightarrow S(\Gamma_{A'})$ which fits into the diagram below on the left. This induces the commuting digram in the middle, which gives rise to an isomorphism $C'(A, A) \rightarrow C'(A', A')$ that fits into the commuting diagram on the right, whose

vertical maps can be both covering maps or both modified retractions.

$$\begin{array}{ccccc}
A & \longrightarrow & A' & C(A, S(\Gamma_A)) \twoheadrightarrow & C(A', S(\Gamma_{A'})) & C'(A, A) \twoheadrightarrow & C'(A', A') \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
S(\Gamma_A) & \twoheadrightarrow & S(\Gamma_{A'}) & S(\Gamma_A) \longrightarrow & S(\Gamma_{A'}) & A \longrightarrow & A'
\end{array}$$

Actually, there is an injective homomorphism from the group of label-preserving automorphisms of A to the group of label-preserving automorphisms of $C'(A, A)$.

7. CONSTRUCTION OF THE FINITE COVER

7.1. A graph of spaces. Throughout this section, Γ will be a finite simplicial graph without induced 4-cycles. And H is a group acting on the blow-up building $Y(\Gamma)$ geometrically by label-preserving automorphisms. Our goal is to show H has a finite index torsion free subgroup $H' \leq H$ such that $Y(\Gamma)/H'$ is a special cube complex. This can be reduced to the following claim.

Claim 7.1. For every vertex $u \in \Gamma$, there exists a finite index torsion free subgroup $\bar{H} \leq H$ such that for any vertex $\bar{u} \in \mathcal{P}(\Gamma)$ labelled by u , the factor action (see Section 4.1) $\rho_{\bar{u}} : \bar{H}_{\bar{u}} \curvearrowright \mathbb{Z}_{\bar{u}}$ is conjugate to an action by translations.

For each vertex of Γ , we find a finite index subgroup of H as in the above claim. Let H' be the intersection of all these subgroups. Then H' satisfies condition (1) of Lemma 4.12, hence $Y(\Gamma)/H'$ is a special cube complex.

We will induct on the number of vertices in Γ , and assume the above claim is true for graphs with $\leq n-1$ vertices. Let Γ be a graph of n vertices without induced 4-cycles. Note that any subgraph of Γ does not have induced 4-cycles.

Pick vertex $u \in \Gamma$. Let $\Lambda \subset \Gamma$ be the induced subgraph spanned by vertices in $\Gamma \setminus \{u\}$ and let $\Lambda_u \subset \Lambda$ be the induced subgraph spanned by vertices adjacent to u (it is possible that $\Lambda_u = \emptyset$ or $\Lambda_u = \Lambda$). By Lemma 5.2, we assume H acts on $Y(\Gamma)$ without inversions. Moreover, let \mathcal{H}_u , T and the action $H \curvearrowright T$ be as in Lemma 5.2. Let $K = Y(\Gamma)/H$. We label the vertices and edges of K as in Section 5.1.

There is a restriction quotient map $q : Y(\Gamma) \rightarrow T$ (see [19, Section 2.3]). In other words, q is the cubical map that collapses every edge of $Y(\Gamma)$ which is not labelled by u to a point. Note that q is H -equivariant. Moreover, q maps each u -component isometrically into T , whose image is called a u -component of T . The collection of u -components in T covers T , and these u -components are permuted under the H -action. We define the *tips* of T to be union of the tips of u -components in T (note that each u -component is a branched line). The sets of tips of T is H -invariant. If $x \in T$ is not a tip, then there is a unique u -component that contains x . Moreover, $q^{-1}(x)$ is isometric to some Λ_u -component of $Y(\Gamma)$. If $x \in T$ is a tip, then $q^{-1}(x)$ is a standard Λ -component of $Y(\Gamma)$.

Let $\mathcal{G} = T/H$ be the quotient graph. We define *tips* and *u-components* in \mathcal{G} to be the images of tips and u -components in T . Here is an alternative characterization of u -components of \mathcal{G} . Pick a vertex $\bar{u} \in \mathcal{P}(\Gamma)$ labelled by u and let $\beta_{P_{\bar{u}}} = \beta_{\bar{u}} \times \beta_{\bar{u}}^\perp$ be as in Lemma 3.9. We identify $\beta_{\bar{u}}$ with its image in T under $Y(\Gamma) \rightarrow T$. Note that if $h \in H$ satisfies that $h(\beta_{\bar{u}}) \cap \beta_{\bar{u}}$ contains a point which is not a tip of $\beta_{\bar{u}}$, then $h(\beta_{\bar{u}}) = \beta_{\bar{u}}$. Thus there is an embedding $\beta_{\bar{u}}/\text{Stab}(\beta_{\bar{u}}) \rightarrow \mathcal{G}$, whose image is a u -component. Note that $\text{Stab}(\beta_{\bar{u}}) = \text{Stab}(\beta_{P_{\bar{u}}})$.

Let E be an Eilenberg–MacLane space for H . Then the diagonal action $H \curvearrowright Y(\Gamma) \times E$ is free. Moreover, there is an H -equivariant projection $Y(\Gamma) \times E \rightarrow Y(\Gamma)$,

which descends to a projection $p : \mathcal{K} = (Y(\Gamma) \times E)/H \rightarrow K$. Note that for any $x \in K$ and $y \in Y(\Gamma)$ which is mapped to x under $Y(\Gamma) \rightarrow K$, $\pi_1(p^{-1}(x))$ is isomorphic to the stabilizer of y . We can view K as a developable complex of groups and \mathcal{K} as the corresponding complex of spaces (see [15, Chapter III.C]).

The above map $q : Y(\Gamma) \rightarrow T$ induces maps $\pi : K \rightarrow \mathcal{G}$ and $\bar{\pi} : \mathcal{K} \rightarrow \mathcal{G}$. These maps induce graph of spaces decompositions

$$K = \left(\bigsqcup_{v \in \text{Vertex}(\mathcal{G})} K_v \sqcup \bigsqcup_{e \in \text{Edge}(\mathcal{G})} (N_e \times [0, 1]) \right) / \sim$$

and

$$\mathcal{K} = \left(\bigsqcup_{v \in \text{Vertex}(\mathcal{G})} \mathcal{K}_v \sqcup \bigsqcup_{e \in \text{Edge}(\mathcal{G})} (\mathcal{N}_e \times [0, 1]) \right) / \sim.$$

Note that $\mathcal{K}_v = p^{-1}(K_v)$ and $\mathcal{N}_e = p^{-1}(N_e)$. There is a 1-1 correspondence between covers of \mathcal{K} and orbifold covers of K , and these covers have induced graph of spaces structures.

Let $\{A(v, i)\}_{i \in I_v}$ be the collection of images of boundary morphisms inside K_v . Then each $A(v, i)$ is a Λ_u -component in K . This component is standard if and only if v is a tip. Let $N(v, i)$ and $\partial_{v, i} : N(v, i) \rightarrow A(v, i)$ be the associated edge space and boundary morphism. It is possible that $N(v, i) \neq N(v, j)$, but $A(v, i) = A(v, j)$. However, this can not happen when v is a tip, since in such case for each vertex in $A(v, i)$, there is only one edge labelled by u which contains this vertex. Note that each $\partial_{v, i}$ preserves edge-labellings. We define $\{\mathcal{A}(v, i)\}_{i \in I_v}$ and $\bar{\partial}_{v, i} : \mathcal{N}(v, i) \rightarrow \mathcal{A}(v, i)$ in a similar way. Each $\bar{\partial}_{v, i}$ is a covering map of finite degree. If v is a tip, then $\bar{\partial}_{v, i}$ is a homeomorphism; if v is not a tip, then $A(v, i) = K_v$ and $\mathcal{A}(v, i) = \mathcal{K}_v$.

For subgraph $\Lambda_1 \subset \Lambda$, a Λ_1 -component $\mathcal{K}_1 \subset \mathcal{K}$ is the inverse image of a Λ_1 -component of K under $p : \mathcal{K} \rightarrow K$. The *parallel set* of \mathcal{K}_1 is the unique Λ_2 -component that contains \mathcal{K}_1 , where $\Lambda_2 = St(\Lambda_1)$ (Definition 2.6). Two Λ_1 -components of \mathcal{K} are *parallel* if they are in the same Λ_2 -component of \mathcal{K} . We can define parallel sets and parallelism between Λ_1 -components of K in a similar way.

The image of every $St(u)$ -component in \mathcal{K} under the map $\bar{\pi} : \mathcal{K} \rightarrow \mathcal{G}$ is a u -component in \mathcal{G} . Thus $\bar{\pi}$ induces a 1-1 correspondence between $St(u)$ -components in \mathcal{K} and u -components in \mathcal{G} . If a u -component in \mathcal{G} contains a circle C , then $\bar{\pi}^{-1}(C)$ is a bundle over C , whose fibres are homeomorphic to some Λ_u -component of \mathcal{K} . The map $\bar{\pi}$ also induces a 1-1 correspondence between Λ -components in \mathcal{K} and tips in \mathcal{G} .

Since Γ contains no induced 4-cycle, when Λ_u is not a clique, every $St(u)$ -component in \mathcal{K} is the parallel set of a Λ_u -component in \mathcal{K} . We divide the proof of Claim 7.1 into the case when Λ_u is a clique, and the case when Λ_u is not a clique. Also we make the additional assumption that $\Gamma \neq St(u)$, since the $\Gamma = St(u)$ case of the claim follows from Lemma 4.10 and Lemma 4.7. When $\Gamma \neq St(u)$, all $St(u)$ -components and Λ -components are standard (Lemma 3.17 (2)) and they intersect along standard Λ_u -components (Lemma 5.6).

Recall that by induction hypothesis, if $\Gamma' \subset \Gamma$ is a induced subgraph that does not contain all vertices of Γ , then each Γ' -component has a finite sheet torsion free special cover. This in particular applies to Λ, Λ_u and $St(u)$.

To simply notation, we will view K and Γ' -components of K ($\Gamma' \subset \Gamma$ is a subgraph) as developable complexes of groups, and work with their orbifold covers. This is equivalent to working with \mathcal{K} and Γ' -components of \mathcal{K} , and using the usual

covering space theory. The fundamental group of a Γ' -component of K is understood to be the orbifold fundamental group of this component, which is isomorphic to the usual fundamental group of the corresponding Γ' -component in \mathcal{K} .

For vertex $v \in \Gamma$, a v -edge in K is an edge labelled by v and a v -circle in K is a core circle made of v -edges. We record the following observation.

Lemma 7.2. *Suppose $\{K_i\}_{i=1}^n$ are finite covers of K such that there exists a torsion free regular cover $\bar{K} \rightarrow K$ such that each K_i factors through \bar{K} . Let K' be the smallest regular cover of K which factors through each K_i . Pick vertex $w \in \Gamma$. Suppose ℓ is an integer such that for each i and each w -circle in K_i , its length divides ℓ . Then the length of each w -circle divides ℓ .*

Proof. Let H_i and \bar{H} be the subgroups of H corresponding to K_i and \bar{K} . Note that the action $\bar{H} \curvearrowright Y(\Gamma)$ is free, and $hH_ih^{-1} \leq \bar{H}$ for each i and $h \in H$. Thus it suffices to show for each line $L \subset Y(\Gamma)$ made of w -edges, each i and each $h \in H$, the translation length of the generator of $\text{Stab}_{hH_ih^{-1}}(L)$ divides ℓ . But this follows from the assumption. \square

Again, the above lemma may not hold if we do not assume K_i factors through \bar{K} . See the example after Lemma 5.7.

7.2. Matching finite covers of vertex spaces and edge spaces. The space $A(v, i)$ is called a *gate* if v is a tip. Our strategy is to construct suitable finite sheet covers for each Λ -component and $St(u)$ -component of K , and glue them together along the elevations of gates.

Lemma 7.3. *There exists a collection \mathcal{C} of finite sheet regular special covers with trivial holonomy, one for each Λ -component and $St(u)$ -component in K , such that for each gate $A(v, i)$, there exists a torsion free regular cover $A_r(v, i)$ such that each possible elevation of $A(v, i)$ to some element of \mathcal{C} factors through $A_r(v, i)$.*

Proof. For each Λ -component and $St(u)$ -component of K , we find a finite sheet special cover which is regular and has trivial holonomy. This is possible by Lemma 5.16 and Lemma 5.10. We denote the resulting collection of covers by \mathcal{C}' . For each gate $A(v, i)$, let $A_r(v, i)$ be the smallest regular cover of $A(v, i)$ which factors through each possible elevation of $A(v, i)$ to elements in \mathcal{C}' . Pick an element $K'_v \in \mathcal{C}'$ which covers a Λ -component $K_v \subset K$. Let K''_v be the smallest regular cover of K_v such that for each gate $A(v, i)$ in K_v , K''_v factors through each component of the canonical completion $C(A_r(v, i), K'_v)$. We replace K'_v by K''_v and replace other elements of \mathcal{C}' in a similar way to obtain a collection \mathcal{C}'' . Moreover, we replace each element in \mathcal{C}'' by a further cover which is regular and has trivial holonomy, using Lemma 5.16 and Lemma 5.10. The resulting collection has the required properties. \square

7.2.1. Case 1: Λ_u is a clique. We first look at the case $\Lambda_u = \emptyset$. It follows from the induction hypothesis that each Λ -component of K has a finite torsion free cover. Moreover, each $St(u)$ -component (or equivalently, u -component) of K has a finite torsion free cover whose fundamental group is \mathbb{Z} . One can glue together suitable many copies of these covers to form a finite cover of K . This cover satisfies the requirement in Claim 7.1. It is torsion free, since each of its vertex spaces and edge spaces is torsion free.

Now we assume Λ_u is a non-empty clique. Let \mathcal{C} be the collection of covers as in Lemma 7.3 and let ℓ be a positive integer such that the length of each core circle (i.e. circle made of core edges) in every element of \mathcal{C} divides ℓ . It follows from Lemma 7.4 below that there exist suitable further finite covers of elements in \mathcal{C} such that we can glue together suitable many copies of them along standard Λ_u -components to form a finite cover of K , which satisfies Claim 7.1 (note that two covers of the a Λ_u -component $L \subset K$ may not be isomorphic as covering spaces of L even when they are the same ℓ -branched torus, however, this is true if they factors through a common torsion free regular cover of L).

Lemma 7.4. *Suppose $K(\Gamma)$ has a finite regular cover $\bar{K}(\Gamma)$ which is special. Let ℓ be a positive integer such that for each vertex $u \in \Gamma$, the length of each u -circle in $\bar{K}(\Gamma)$ divides ℓ . Then $K(\Gamma)$ has a finite index regular cover K' such that for each clique $\Delta \subset \Gamma$, each standard Δ -component of K' is an ℓ -branched torus, i.e. it is isomorphic as cube complexes to a product of branched circles whose core circle have length $= \ell$.*

Proof. By Lemma 5.14, we can assume $\bar{K}(\Gamma)$ is a finite cover of the Salvetti complex $S(\Gamma)$ (note that all u -circles in $\bar{K}(\Gamma)$ has length $= \ell$ if and only if the same is true for the reduction of $\bar{K}(\Gamma)$). For a clique $\Delta \in \Gamma$, let $T_\Delta \rightarrow S(\Delta)$ be the ℓ -branched torus which covers $S(\Delta)$. Define $\hat{K}(\Gamma)$ to be the pull-back as follows.

$$\begin{array}{ccc} \hat{K}(\Gamma) & \longrightarrow & C(T_\Delta, S(\Gamma)) \\ \downarrow & & \downarrow \\ \bar{K}(\Gamma) & \longrightarrow & S(\Gamma) \end{array}$$

Since T_Δ factors through each Δ -component of $\bar{K}(\Gamma)$, each Δ -component of $\hat{K}(\Gamma)$ is isomorphic to T_Δ . We claim that (1) for each $u \in \Gamma$, the length of each u -circle in $\hat{K}(\Gamma)$ divides ℓ ; (2) if we already know there is a clique $\Delta' \subset \Gamma$ such that each Δ' -component of $\bar{K}(\Gamma)$ is isomorphic to $T_{\Delta'}$, then the same is true for Δ' -component of $\hat{K}(\Gamma)$. To see (1), note that each u -circle in $S(\Gamma)$ has length $= 1$, and a u -circle in $C(T_\Delta, S(\Gamma))$ has length equal to either ℓ or 1. (2) follows from (1) and the fact that $C(T_\Delta, S(\Gamma))$ has trivial holonomy (Lemma 5.15).

Now we can repeat the above process for each clique in Γ to obtain a finite cover $\hat{K}(\Gamma)$ of $\bar{K}(\Gamma)$ such that for each clique $\Delta \subset \Gamma$, each Δ -component in $\hat{K}(\Gamma)$ is isomorphic to T_Δ . Let K' to be the smallest regular cover of $K(\Gamma)$ that factors through $\hat{K}(\Gamma)$. Then K' has the required property by Lemma 7.2. \square

7.2.2. Case 2: Λ_u is not a clique. Pick a $St(u)$ -component $L \subset K$. Then L can be viewed as a graph of spaces over a u -component $\mathcal{G}_L \subset \mathcal{G}$ whose boundary morphisms are covering maps of finite degree. A finite sheet cover L' of L is *admissible* if

- (1) there exists a torsion free regular cover $\bar{L} \rightarrow L$ such that each component of L' factors through \bar{L} ;
- (2) each component of L' is a trivial bundle over a branched circle (we only require the bundle is topologically trivial, however, it may have non-trivial u -holonomy).

The existence of such cover follows from Lemma 4.10 and the induction hypothesis. It follows from Lemma 4.12 that to show Claim 7.1, it suffices to find a finite torsion free cover of K such that each of its $St(u)$ -component is admissible.

Observation 7.5. *Let M be an admissible cover of L with trivial u -holonomy.*

- (1) *The smallest regular cover of L which factors through each component of M also has trivial u -holonomy.*
- (2) *Each component of $C'(M, M)$ has trivial u -holonomy.*
- (3) *Let $M' \subset M$ be a vertex space or edge space. Then the inverse image of M' under the covering $C'(M, M) \rightarrow M$ is naturally isomorphic to $C(M', M')$.*

Here (1) follows from Lemma 5.7. (2) follows from the argument in Lemma 5.15. Since M has trivial u -holonomy, it is a product. Thus M' is wall-injective in M and (3) follows from Lemma 6.11.

A collection Σ of (not necessarily connected) finite sheet covers, one for each gate, is called *admissible* if there exists a collection Φ of admissible covers, one for each $St(u)$ -component in K , such that for each gate, its inverse image in the element of Φ that covers this gate is a disjoint union of spaces, each of which is isomorphic to the element in Σ that covers this gate, in the sense of covering spaces. Σ has *trivial u -holonomy* if we can choose the collection Φ such that each of its element has trivial u -holonomy. Σ is *regular admissible* if each element in Σ is connected and each element in Φ is a regular cover. In generally, being a regular admissible collection is stronger than being an admissible collection of regular coverings.

In the rest of this section, we will use a modified version of the argument in [39, Section 6]. The reader could also consult [64, Page 51-52] for a pictorial illustration of the strategy in the malnormal case.

Lemma 7.6. *For each tip $v \in \mathcal{G}$ and each gate $A(v, i)$ in K_v , we can find a finite sheet base pointed cover $K(v, i) \rightarrow K_v$ (the base point of K_v is in $A(v, i)$, and we allow the base point to change for different gates in K_v) such that the following properties hold.*

- (1) *Each $K(v, i)$ is torsion free and special.*
- (2) *The based elevation $A'(v, i) \rightarrow K(v, i)$ of $A(v, i) \rightarrow K_v$ is wall-injective.*
- (3) *Each $A'(v, i)$ has trivial holonomy. The collection of all such coverings $A'(v, i) \rightarrow A(v, i)$ is a regular admissible collection with trivial u -holonomy.*
- (4) *For each gate $A(v, i)$, there exists a torsion free regular cover $A_r(v, i)$ such that each elevation of $A(v, i)$ to $K(v, j)$ (it is possible that $j \neq i$) factors through $A_r(v, i)$.*
- (5) *$WProj_{K(v, i)}(B \rightarrow A'(v, i))$ is a disjoint union of branched torus and isolated points for any Λ_u -component $B \subset K(v, i)$ different from $A'(v, i)$.*

Proof. Let \mathcal{C} be the collection in Lemma 7.3 and let $L' \in \mathcal{C}$ be an element which covers a $St(u)$ -component $L \subset K$. Pick a gate $A(v, i) \subset L$. Let $A_1(v, i) \rightarrow A(v, i)$ be a finite cover which factors through each elevation of $A(v, i)$ to L' and K'_v (K'_v is the element \mathcal{C} that covers K_v). We apply Corollary 6.5 to $A_1(v, i) \rightarrow K'_v$ to obtain a finite cover $A_2(v, i) \rightarrow A_1(v, i)$ such that any further finite cover of $A_2(v, i)$ with trivial holonomy will satisfy the conclusion of Corollary 6.5. Let $L'' \rightarrow L$ be a regular special cover with trivial holonomy such that for each gate $A(v, i) \subset L$, L'' factors through each component of the canonical completion $C(A_2(v, i), L')$. We choose $A'(v, i)$ to be an elevation of $A(v, i)$ to L'' (since L'' is regular, all elevations are the same). (3) is clear since L'' has trivial holonomy. Since $A'(v, i)$ factors through $A_2(v, i)$, by Corollary 6.5, we can find finite cover $K(v, i) \rightarrow K'_v$ which satisfies (1), (2) and (5). (4) follows from Lemma 7.3. \square

For each $A'(v, i)$, we form the modified completions $C'(A'(v, i), A'(v, i))$ and $C'(A'(v, i), K(v, i))$, and denote them by $C'_{v,i}(A', A')$ and $C'_{v,i}(A', K)$ for simplicity. By Lemma 6.11, there is a canonical inclusion $C'_{v,i}(A', A') \rightarrow C'_{v,i}(A', K)$. Moreover, it follows from Observation 7.5 (3) that the collection of all $C'_{v,i}(A', A')$'s is admissible.

For vertex $a \in \Gamma$, we define ℓ_a to be the least common multiple of the lengths of all a -circles in the collection $\{C'_{v,i}(A', K)\}_{v \in \text{Tip}(\mathcal{G}), i \in I_v}$.

Let $\pi : C'_{v,i}(A', K) \rightarrow K(v, i)$ be the covering map and let $r' : C'_{v,i}(A', K) \rightarrow A'(v, i)$ be the modified retraction. Pick a Λ_u -component $A' \subset C'_{v,i}(A', K)$, by Corollary 6.5 and Lemma 6.7, if $\pi(A') \neq A'(v, i)$, then $r'(A')$ is contained in a branched torus. Let T be the smallest branched torus containing $r'(A')$ and let $\{v_i\}_{i=1}^n$ be the label of edges in T . Since $A'(v, i)$ has trivial holonomy, T has a cubical product decomposition $T = \prod_{i=1}^n C_i$ where each C_i is a v_i -component in T . Let C'_i be a cover of C_i of degree $= \ell_{v_i}/n_i$ where n_i is the length of the core of C_i , and let $T' = \prod_{i=1}^n C'_i$. We define the *shadow* of A' (when $\pi(A') \neq A'(v, i)$) to be one connected component of the pull-back A'_S as follows.

$$\begin{array}{ccc} A'_S & \longrightarrow & T' \\ \downarrow & & \downarrow \\ A' & \xrightarrow{r'} & T \end{array}$$

It is possible that both T' and T are one point, in which case $A'_S = A'$. The definition of shadow does not depend on the choice of components in A'_S , since all of them give the same regular cover of A' . We record the following two observations.

Remark 7.7. (1) Let $T_1 \rightarrow T$ be any cover such that for each i , the length of the core of v_i -components in T_1 divides ℓ_{v_i} . Then T' factors through T_1 .

(2) Let C' be a circle made of core edges in A' . By Corollary 6.10, either $r'(C')$ is a point, in which case all inverse images of C' in A'_S are circles which have the same length as C' ; either $r'|_{C'}$ is an isomorphism, in which case the inverse image of C' in A'_S is a circle of length ℓ_a where a is the label of edges in C' .

Let $A(v, i)$ be a gate. Let $\dot{A}(v, i) \rightarrow A(v, i)$ be the smallest regular cover that factors through each elevation of $A(v, i)$ to $C'_{v,j}(A', K)$ and the shadow of this elevation (if exists), here (v, j) ranges over all pairs such that $A(v, j) \subset K_v$.

Let L be the $St(u)$ -component that contains $A(v, i)$ and let L' be the cover of L induced by the admissible collection of $A'(v, i)$'s. Recall that L' has trivial u -holonomy, hence it is isomorphic to a product of $A'(v, i)$ with a u -component in L' . Then $\dot{A}(v, i) \rightarrow A'(v, i)$ induces a cover of L' , which we denote by $\dot{L}(v, i)$. Let \bar{L} be the smallest regular cover of L which factors through each $\dot{L}(v', i')$, where (v', i') is a pair such that $A(v', i')$ is a gate in L . Then \bar{L} has trivial u -holonomy by Lemma 5.7. Let $\bar{A}(v, i) \rightarrow A(v, i)$ be the regular cover induced by $\bar{L} \rightarrow L$. Then the collection of all $\bar{A}(v, i)$'s is regular admissible with trivial u -holonomy.

It follow from Remark 7.7 (2) that the length of each a -circle in the spaces involved in the construction of $\bar{A}(v, i)$ divides ℓ_a , thus the lemma below follows from Lemma 7.2 and Lemma 7.6 (4).

Lemma 7.8. *Let $a \in \Lambda_u$ be a vertex. Then the length of any a -circle in $\bar{A}(v, i)$ divides ℓ_a .*

For each pair (v, i) , we define the pull-back $\overline{C'_{v,i}(A', K)}$ and $\overline{C'_{v,i}(A', A')}$ as follows.

$$\begin{array}{ccc}
\overline{C'_{v,i}(A', K)} & \longrightarrow & \bar{A}(v, i) \\
\downarrow & & \downarrow \\
C'_{v,i}(A', K) & \longrightarrow & A'(v, i)
\end{array}
\quad
\begin{array}{ccc}
\overline{C'_{v,i}(A', A')} & \longrightarrow & \bar{A}(v, i) \\
\downarrow & & \downarrow \\
C'_{v,i}(A', A') & \longrightarrow & A'(v, i)
\end{array}$$

Then one deduce from Lemma 6.11 that there is a naturally defined embedding $i_1 : \overline{C'_{v,i}(A', A')} \rightarrow \overline{C'_{v,i}(A', K)}$ which fits into the following commuting diagram. Moreover, $f^{-1}(i_2(C'_{v,i}(A', A'))) = i_1(\overline{C'_{v,i}(A', A')})$.

$$\begin{array}{ccc}
\overline{C'_{v,i}(A', A')} & \xrightarrow{i_1} & \overline{C'_{v,i}(A', K)} \\
\downarrow & & \downarrow f \\
C'_{v,i}(A', A') & \xrightarrow{i_2} & C'_{v,i}(A', K)
\end{array}$$

We claim the collection of $\overline{C'_{v,i}(A', A')}$'s is admissible. Pick a $St(u)$ -component, and let M_1, M_2 and M_3 be the covers of this component induced by the collection of $C'_{v,i}(A', A')$'s, $A'(v, i)$'s and $\bar{A}(v, i)$'s respectively. Recall that M_2 and M_3 have trivial u -holonomy, and the same is true for M_1 by Observation 7.5 (2). By Observation 7.5 (3), the modified retraction $M_1 \rightarrow M_2$ is compatible with the retraction $C'_{v,i}(A', A') \rightarrow A'(v, i)$. Then the claim follows by considering the pull-back of the covering $M_3 \rightarrow M_2$ under the retraction $M_1 \rightarrow M_2$.

Lemma 7.9. *Let $A(v, i)$ be a gate. Suppose \bar{A} is either (1) an elevation of $A(v, i)$ to $\overline{C'_{v,i}(A', K)}$ such that the image of \bar{A} under the covering map $\overline{C'_{v,i}(A', K)} \rightarrow K(v, i)$ is distinct from $A'(v, i)$; or (2) an elevation of $A(v, i)$ to $\overline{C'_{v,j}(A', K)}$ with $i \neq j$. Then $\bar{A}(v, i)$ factors through \bar{A} .*

Proof. We prove case (2). The proof of case (1) is similar. The following diagram indicate the history of \bar{A} :

$$\begin{array}{ccccccc}
\bar{A} & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A(v, i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\overline{C'_{v,j}(A', K)} & \longrightarrow & C'_{v,j}(A', K) & \longrightarrow & K(v, j) & \longrightarrow & K_v
\end{array}$$

Since $i \neq j$, $A_1 \neq A'(v, j)$. By Theorem 6.5 and Lemma 6.7, $r'(A_2)$ is contained in a branched torus. Let T be the smallest such branched torus. The case T is a point is clear. Otherwise, let \bar{T} be any lift of T in $\bar{A}(v, j)$. It follows from Lemma 7.8 that the length of any a -circle in \bar{T} divides ℓ_a . It follows from Remark 7.7(1) that the shadow of A_2 factors through \bar{A} , thus $\bar{A}(v, i)$ factors through \bar{A} . \square

For each tip $v \in \mathcal{G}$, let $\mathring{K}(v, i)$ be the smallest regular cover of K_v factoring through each component of $\overline{C'_{v,i}(A', K)}$ and let \mathring{K}_v be the smallest regular cover of K_v factoring through each $\mathring{K}(v, i)$. Let $\mathring{A}(v, i)$ be the smallest regular cover of $A(v, i)$ factoring through each component of $\overline{C'_{v,i}(A', A')}$.

Lemma 7.10.

- (1) *The collection $\{\mathring{A}(v, i)\}_{v \in Tip(\mathcal{G}), i \in I_v}$ is regular admissible.*
- (2) *Each elevation of $A(v, i)$ to \mathring{K}_v is isomorphic to $\mathring{A}(v, i)$ as covering spaces of $A(v, i)$.*

Proof. Since the collection of $\overline{C'_{v,i}(A', A')}$'s is admissible, (1) follows. Let \mathring{A} be a lift of $A(v, i)$ to \mathring{K}_v . By Lemma 6.11, the lift of $A(v, i)$ to $C'_{v,i}(A', K')$ could be any component of $C'_{v,i}(A', A')$, hence the lift of $A(v, i)$ to $\overline{C'_{v,j}(A', K)}$ could be any component of $\overline{C'_{v,j}(A', A')}$. Thus \mathring{A} factors through $\mathring{A}(v, i)$. On the other hand, Lemma 7.9 implies that $\mathring{A}(v, i)$ factors through each elevation of $A(v, i)$ to $\overline{C'_{v,j}(A', K)}$ for each j . Thus $\mathring{A}(v, i)$ factors through \mathring{A} . \square

The collection in Lemma 7.10 (1) induces a finite sheet cover for each $St(u)$ -component in K . Lemma 7.10 (2) enables us to glue suitable many copies of these covers together with copies of \mathring{K}_v 's to form a finite sheet cover of K , which satisfies Claim 7.1. Thus we have proved the following result.

Corollary 7.11. *Suppose Γ is a graph without induced 4-cycle. Let $H \curvearrowright Y(\Gamma)$ be a group acting geometrically by label-preserving automorphisms on a blow-up building $Y(\Gamma)$. Then H has a finite index subgroup $H' \leq H$ such that $Y(\Gamma)/H'$ is a special cube complex.*

The next result follows from Definition 4.9, Lemma 4.6 and Corollary 7.11:

Theorem 7.12. *Suppose Γ is a graph such that*

- (1) Γ is star-rigid;
- (2) Γ does not contain induced 4-cycle;
- (3) $\text{Out}(G(\Gamma))$ is finite.

Then any finite generated group quasi-isometric to $G(\Gamma)$ is commensurable to $G(\Gamma)$.

8. UNIFORM LATTICES IN $\text{Aut}(X(\Gamma))$

In this section we study groups acting geometrically on $X(\Gamma)$.

8.1. Admissible edge labellings of $X(\Gamma)$. Recall that we label circles in the 1-skeleton of the Salvetti complex $S(\Gamma)$ by vertices in Γ . This induces a $G(\Gamma)$ -invariant edge-labelling on the universal cover $X(\Gamma)$. Let x be a vertex in $S(\Gamma)$ or $X(\Gamma)$. Then each vertex in the link of x comes from an edge, hence inherits a label. Let $F(\Gamma)$ be the flag complex of Γ . Then there is a projection map which preserves the label of vertices.

$$(8.1) \quad p_x : Lk(x, X(\Gamma)) \rightarrow F(\Gamma)$$

See Definition 2.6 for $Lk(x, X(\Gamma))$. Note that for each vertex $v \in F(\Gamma)$, $p_x^{-1}(v)$ is a pair of vertices. We also pick an orientation for each edge in $S(\Gamma)$, and lift them to orientations of edges in $X(\Gamma)$ which is $G(\Gamma)$ -invariant. We fix a $G(\Gamma)$ -invariant labelling and a $G(\Gamma)$ -invariant orientation of edges in $X(\Gamma)$, and call them the *reference labelling* and *reference orientation*.

Throughout this section, Γ will be a graph without induced 4-cycle. A vertex $v \in \Gamma$ is of *type I* if there exists a vertex $w \in \Gamma$ not adjacent to v such that $lk(v) \subset St(w)$. Otherwise v is of *type II*. Two vertices v_1 and v_2 of type I are *equivalent* if they are adjacent. Now we verify this is indeed an equivalence relationship. Since Γ has no induced 4-cycle, the link of each vertex of type I has to be a (possibly empty) clique, thus v_1 and v_2 are adjacent if and only if $St(v_1) = St(v_2)$.

Given vertex v of type I, let $[v]$ be the clique in Γ spanned by type I vertices which are equivalent to v and let $lk([v])$ be the clique spanned by vertices in $St(v) \setminus [v]$. The definition of $lk([v])$ does not depend on the choice of representative in v . Any

vertex in $lk([v])$ is of type II. Since the closed star of each vertex in $[v]$ is a clique, a vertex $w \in \Gamma \setminus [v]$ is in $lk([v])$ if and only if w is adjacent to one vertex in $[v]$ if and only if w is adjacent to each vertex in $[v]$.

Let Γ_i be the induced subgraph of Γ spanned by vertices of type i . Then Γ_1 is a disjoint union of cliques, one for each equivalent class of vertices of type I. Let $\Delta \subset \Gamma_2$ be a clique. Let $\Gamma_{1,\Delta} \subset \Gamma_1$ be the union of all $[v]$'s such that $lk([v]) = \Delta$. We allow $\Delta = \emptyset$. Note that $\Gamma_{1,\emptyset}$ is always a (possibly empty) discrete graph. Also it follows from the definition that for cliques $\Delta_1, \Delta_2 \subset \Gamma_2$, if $\Delta_1 \neq \Delta_2$, then $\Gamma_{1,\Delta_1} \cap \Gamma_{1,\Delta_2} = \emptyset$.

Lemma 8.2. *Pick $\alpha \in \text{Aut}(X(\Gamma))$ and pick vertex $v \in \Gamma_2$.*

- (1) *For any v -component $\ell \subset X(\Gamma)$, $\alpha(\ell)$ is a v' -component for vertex $v' \in \Gamma_2$.*
- (2) *Let $\Delta \subset \Gamma_2$ be a clique. Suppose α sends a Δ -component to a Δ' -component for $\Delta' \subset \Gamma_2$ (this follows from (1)). Then $\Gamma_{1,\Delta}$ is isomorphic to $\Gamma_{1,\Delta'}$.*

Proof. Let $P_\ell = \ell \times \ell^\perp$ be the $St(v)$ -component containing ℓ with its natural splitting. Then $\alpha(P_\ell) = \alpha(\ell) \times \alpha(\ell^\perp)$. Let L_1 and L_2 be the collection of labels of edges in $\alpha(\ell)$ and $\alpha(\ell^\perp)$ respectively. To prove (1), it suffices to show L_1 is made of one vertex. Suppose the contrary is true. Let K be the L_1 -component that contains $\alpha(\ell)$. Since each vertex in L_1 is adjacent to every vertex in L_2 , there exists a copy of $K \times \alpha(\ell^\perp)$ in $X(\Gamma)$ which contains $\alpha(P_\ell)$. Then $\ell \times \ell^\perp \subset \alpha^{-1}(K) \times \ell^\perp$. Thus the label of each edge in $\alpha^{-1}(K)$ is adjacent to every vertex in $lk(v)$. Since $\alpha^{-1}(K)$ is not isometric to a line, it has an edge whose label (denoted by w) is different from v . Thus $lk(v) \subset St(w)$ and $w \notin lk(v)$, which contradicts that v is of type II.

Suppose edges of $\alpha(\ell)$ are labelled by v' . Now we show $v' \in \Gamma_2$. If the contrary is true, then there is a vertex $w' \in \Gamma \setminus St(v')$ such that $lk(v') \subset lk(w')$. Let K be the $\{v', w'\}$ -component containing $\alpha(\ell)$. Since $L_2 \subset lk(v')$, $\alpha(\ell) \times \alpha(\ell^\perp) \subset K \times \alpha(\ell^\perp) \subset X(\Gamma)$. Now we can reach a contradiction as in the previous paragraph.

Now we prove (2). The case $\Delta = \Delta' = \emptyset$ is trivial. We assume $\Delta \neq \emptyset$. Let F and $F' = \alpha(F)$ be the Δ -component and Δ' -component as in (2). Pick vertex $x \in F$ and let $x' = \alpha(x)$. Note that α induces an isomorphism $\alpha_x : Lk(x, X(\Gamma)) \rightarrow Lk(x', X(\Gamma'))$. Let p_x and $p_{x'}$ be the maps defined as in (8.1). A vertex v in $Lk(x, X(\Gamma))$ (or $Lk(x', X(\Gamma'))$) is of type I if $p_x(v)$ (or $p_{x'}(v)$) is of type I, otherwise v is of type II. It follows from (1) that α_x induces a bijection between vertices of type II in $Lk(x, X(\Gamma))$ and $Lk(x', X(\Gamma'))$. Thus if we replace type II by type I in the previous sentence, it still holds.

Pick a vertex $u \in Lk(x, X(\Gamma))$ such that $p_x(u) \in \Gamma_{1,\Delta}$. Let e_u be the edge containing x which gives rise to u in the link of x . Note that $u' = \alpha_x(u)$ is a vertex of type I. Moreover, since e_u is orthogonal to F , $e_{u'} = \alpha(e_u)$ is orthogonal to F' . Thus $p_{x'}(u') \in \Gamma_{1,\Delta'_1}$ for some clique Δ'_1 which contains Δ' . We claim $\Delta' = \Delta'_1$. If $\Delta' \subsetneq \Delta'_1$, let F'_1 be the standard flat such that its support is Δ'_1 and it contains F' . Let $F_1 = \alpha^{-1}(F'_1)$. Then F_1 is a standard flat (this follows from (1) since $\Delta'_1 \subset \Gamma_2$), and $F \subsetneq F_1$. Since $e_{u'}$ is orthogonal to F'_1 , e_u is orthogonal to F_1 . Thus $p_x(u) \in \Gamma_{1,\Delta_1}$ for some Δ_1 containing the support of F_1 . This contradicts $p_x(u) \in \Gamma_{1,\Delta}$ since $\Delta_1 \neq \Delta$. Thus the claim holds. Let V (or V') be vertices in $Lk(x, X(\Gamma))$ (or $Lk(x', X(\Gamma'))$) whose p_x -images (or $p_{x'}$ -images) are in $\Gamma_{1,\Delta}$ (or $\Gamma_{1,\Delta'}$). Then the claim implies α_x induces a bijection between V and V' , hence it also induced an isomorphism between the full subcomplexes of $Lk(x, X(\Gamma))$ and $Lk(x', X(\Gamma'))$ spanned by V and V' respectively. However, these two full subcomplexes are isomorphic to the links of

the base points in the Salvetti complexes $S(\Gamma_{1,\Delta})$ and $S(\Gamma_{1,\Delta'})$ respectively. Thus $\Gamma_{1,\Delta}$ and $\Gamma_{1,\Delta'}$ are isomorphic (however, the isomorphism between them may not be induced by α_x). \square

Lemma 8.3. *Let X be a $CAT(0)$ cube complex. Suppose it is possible to label the edges of X by vertices of Γ such that for each vertex $x \in X$, the link of x in X is isomorphic to the link of the base point in $S(\Gamma)$ in a label-preserving way. Then there is a label-preserving isomorphism $X(\Gamma) \cong X$.*

Proof. Pick a vertex $a \in \Gamma$. It follows from the assumption that each a -component in X is a line, which is called an a -line. Moreover, the following are true:

- (1) Each a -line is a convex subcomplex of X .
- (2) Two a -lines l_1 and l_2 are parallel if and only if there exist edges $e_1 \subset l_1$ and $e_2 \subset l_2$ such that e_1 and e_2 are parallel.

We orient each edge of X as follows. For each vertex $a \in \Gamma$, we group the collection of a -lines into parallel classes. In each parallel class, we choose an orientation for one a -line, and extend to other a -lines in the parallel class by parallelism. It follows from (2) of the previous paragraph that if two edges are parallel, then their orientation is compatible with the parallelism.

Pick base vertices $x \in X(\Gamma)$ and $x' \in X$. By looking at the oriented labelling on the edges of X and $X(\Gamma)$, every word in $G(\Gamma)$ corresponds to a unique edge path in $X(\Gamma)$ (or X) starting at x (or at x'), and vice versa. We define a map $f : X(\Gamma)^{(0)} \rightarrow X^{(0)}$ as follows. Pick vertex $y \in X(\Gamma)$ and an edge path ω from x to y , then there is an edge path $\omega' \subset X$ from x' to y' such that ω and ω' correspond to the same word. We define $f(y) = y'$. This definition does not depend on the choice of ω . Actually if we pick two words w_1 and w_2 whose corresponding endpoints are y , then we can obtain w_2 from w_1 by performing the following two basic moves:

- (1) $waa^{-1}w' \rightarrow ww'$.
- (2) $wabw' \rightarrow wbaw'$ when a and b commutes.

However, these two moves do not affect the position of y' . Note that f is surjective and can be extended to a local isometry from $X(\Gamma)$ to X . Thus $X(\Gamma) \cong X$. \square

Definition 8.4. A labelling of edges in $X(\Gamma)$ is *admissible* if it satisfies the assumption of Lemma 8.3.

Corollary 8.5. *Suppose H acts on $X(\Gamma)$ by automorphisms such that it preserves an admissible labelling of $X(\Gamma)$. Then there exists $g \in \text{Aut}(X(\Gamma))$ such that gHg^{-1} preserves the reference labelling of $X(\Gamma)$.*

Lemma 8.6. *Suppose H acts on $X(\Gamma)$ geometrically by automorphisms such that it preserves the reference labelling of $X(\Gamma)$. If Γ does not have induced 4-cycle, then there exist a torsion free finite index subgroup $H' \leq H$ and an element $g \in \text{Aut}(X(\Gamma))$ such that $gH'g^{-1}$ preserves both the reference labelling and the reference orientation. Hence $gH'g^{-1} \leq G(\Gamma)$.*

Proof. By Corollary 7.11, there is a finite index torsion free subgroup $H' \leq H$ be such that $X(\Gamma)/H'$ is special. Lemma 4.12 implies that we can orient each standard geodesic line in a way which is compatible with parallelism and H' -action. The argument in Lemma 8.3 implies that there exists a label-preserving automorphism $g \in \text{Aut}(X(\Gamma))$ which maps this orientation to the reference orientation. Thus the lemma follows. \square

We recall the following result from [4].

Theorem 8.7. *Let T be a uniform tree and let $H_1, H_2 \leq \text{Aut}(T)$ be two uniform lattices (i.e. they act geometrically on T). Then there exists $g \in \text{Aut}(T)$ such that $gH_1g^{-1} \cap H_2$ is of finite index in both H_2 and gH_1g^{-1} .*

Corollary 8.8. *Suppose Γ is a disjoint union of cliques. Let H be a group acting geometrically on $X(\Gamma)$ by automorphisms. Then there exist a finite index torsion free subgroup $H' \leq H$ and an admissible labelling of $X(\Gamma)$ which is invariant under H' .*

Actually in this case H has a finite index subgroup which is isomorphic to a free product of finite generated free Abelian groups [5, Theorem 2]. However, the corollary does not follow directly from this fact.

Proof. We form a partition $\Gamma = \sqcup_{i=1}^k \Gamma_i$ such that each Γ_i is made of cliques of the same size, and cliques in different Γ_i 's have different size. We claim it suffices to prove the corollary for each Γ_i . To see this, suppose $k \geq 2$. Let t_k be the tree of diameter 2 with k vertices of valance one. For each i , we glue $S(\Gamma_i)$ to an endpoint of t_k along the unique vertex in $S(\Gamma_i)$ such that different $S(\Gamma_i)$'s are glued to different endpoints of t_k . Then the resulting space $\bar{S}(\Gamma)$ is homotopic equivalent to $S(\Gamma)$. We pass to the universal cover $\bar{X}(\Gamma)$, and collapse each elevation of $S(\Gamma_i)$ ($1 \leq i \leq k$) in $\bar{X}(\Gamma)$. The resulting space is a tree, which we denote by T .

Since the action $H \curvearrowright X(\Gamma)$ maps Γ_i -components to Γ_i -components, there is an induced action $H \curvearrowright \bar{X}(\Gamma)$, moreover, H permutes the elevations of $S(\Gamma_i)$'s. Thus there is an induced action $H \curvearrowright T$. By passing to a finite index subgroup, we assume H acts on T without inversion. Then there is a graph of spaces decomposition of the orbifold $\bar{X}(\Gamma)/H$ along the graph T/H . The fundamental group of each edge orbifold is finite, and the fundamental group of each vertex orbifold is either finite, or isomorphic to a group acting geometrically on $X(\Gamma_i)$. If the corollary is true for each Γ_i , then we can argue as in Section 7 to pass to suitable torsion free finite sheet covers of each edge space and vertex space, and glue them together to form a finite sheet cover of $\bar{X}(\Gamma)/H$, which gives the required subgroup of H .

Now we look at the case when Γ is a disjoint union of p copies of n -cliques. First we assume $n \geq 2$. By the argument of the previous paragraph (we consider the space obtained by attaching p tori to the valance one vertices of t_p , then H acts on the universal cover of this space), we can assume H is torsion free by passing to a finite index subgroup. Moreover, we can assume $K = X(\Gamma)/H$ is a union of tori (i.e. K does not contain Klein bottles, however, we do not require these tori to be embedded). It suffices to show K has a finite sheet cover which covers $S(\Gamma)$.

We pick a copy of \mathbb{S}^3 with a free \mathbb{Z}/p action. Pick a \mathbb{Z}/p -orbit of \mathbb{S}^3 and we glue a n -torus to \mathbb{S}^3 along each point in the orbit. The resulting space has the same fundamental group as $S(\Gamma)$, and admits a free \mathbb{Z}/p action. Denote the quotient space by $S_p(\Gamma)$. We modify the complex K in a similar way. Namely for each vertex $x \in K$, there are p tori containing x . We replace x by a copy of \mathbb{S}^3 and re-glue those tori along a \mathbb{Z}/p -orbit in \mathbb{S}^3 . The resulting complex K' has the same fundamental group as K . Since K' does not contain Klein bottles, there exists a finite sheet covering map $K' \rightarrow S_p(\Gamma)$, which implies K has a finite sheet cover which covers $S(\Gamma)$. When $n = 1$, it follows from [5, Theorem 2] that H has a finite index torsion free subgroup. The rest follows from Theorem 8.7. Alternatively, it is not hard to modify the above argument such that it also works for $n = 1$. \square

8.2. The conjugation theorem.

Theorem 8.9. *Suppose Γ is a simplicial graph such that*

- (1) Γ is star-rigid;
- (2) Γ does not contain induced 4-cycle.

Let H be a group acting geometrically on $(X(\Gamma))$ by automorphisms. Then H and $G(\Gamma)$ are commensurable. Moreover, let $H_1, H_2 \leq \text{Aut}(X(\Gamma))$ be two uniform lattices. Then there exists $g \in \text{Aut}(X(\Gamma))$ such that $gH_1g^{-1} \cap H_2$ is of finite index in both gH_1g^{-1} and H_2 .

Proof. If $\Gamma_2 = \emptyset$, then Γ is a discrete set and the theorem follows from Corollary 8.8. Now we assume $\Gamma_2 \neq \emptyset$. Pick vertex $v \in \Gamma_2$, it follows from Lemma 8.2 (1) that α sends each standard geodesic in $X(\Gamma)$ labelled by a vertex in Γ_2 to another standard geodesic labelled by a (possibly different) vertex in Γ_2 . Thus the collection of all Γ_2 -components in $X(\Gamma)$ is H -invariant. It follows that the stabilizer of each Γ_2 -component acts cocompactly on itself.

Pick vertex $x \in X(\Gamma)$. Then α induces an isomorphism between the links of x and $\alpha(x)$, which gives rise to an automorphism $\alpha_x : \Gamma_2 \rightarrow \Gamma_2$ by Lemma 8.2 (1). We claim that α_x can be extended to an automorphism of Γ . Recall that Γ_1 is a disjoint union of $\Gamma_{1,\Delta}$ with Δ ranging over cliques (including the empty clique) of Γ_2 . By Lemma 8.2 (2), we can specify an isomorphism $\Gamma_{1,\Delta} \rightarrow \Gamma_{1,\alpha_x(\Delta)}$ for each Δ , and use them to define the extension of α_x as required.

Pick another vertex $y \in X(\Gamma)$ which is adjacent to x along an edge e . We claim $\alpha_x = \alpha_y$. Let $v_e \in \Gamma$ be the label of e . Then $\alpha_x(v) = \alpha_y(v)$ for any $v \in \Gamma_2$ which is adjacent to v_e . Suppose $v_e \notin \Gamma_2$. Then $\alpha_x(lk([v_e])) = \alpha_y(lk([v_e]))$. We can extend α_x and α_y to be automorphisms of Γ which agree on $St(v_e)$ (recall that $St(v_e)$ is a clique spanned by $[v_e]$ and $lk([v_e])$). Thus $\alpha_x = \alpha_y$ by condition (1) of the theorem. Suppose $v_e \in \Gamma_2$ and let $v \in \Gamma_1$ be a vertex adjacent to v_e . Then $lk([v]) \subset \Gamma_2 \cap St(v_e)$. Thus α_x and α_y agree on $lk([v])$. Again we can extend α_x and α_y to automorphisms of Γ which agree on $St(v_e)$ and deduce $\alpha_x = \alpha_y$.

The above claim implies α determines a well-defined element in $\text{Aut}(\Gamma_2)$. Thus we have a homomorphism $H \rightarrow \text{Aut}(\Gamma_2)$. By replacing H by the kernel of this homomorphism, we assume H maps v -components to v -components for $v \in \Gamma_2$. Thus for each clique $\Delta \subset \Gamma_2$, H preserves Δ -components. Then it follows from the proof of Lemma 8.2 (2) that for each edge $e \in X(\Gamma)$ labelled by a vertex in $\Gamma_{1,\Delta}$, the label of $\alpha(e)$ is also inside $\Gamma_{1,\Delta}$. Thus H preserves $\Gamma_{1,\Delta}$ -components.

Let $\{\Delta_i\}_{i=1}^k$ be the collection of cliques in Γ_2 such that $\Gamma_{1,\Delta_i} \neq \emptyset$. Let Λ_i be the induced subgraph spanned by Δ_i and Γ_{1,Δ_i} . Now we define a new cube complex

$$\bar{S}(\Gamma) = (S(\Gamma_2) \sqcup (\sqcup_{i=1}^k S(\Lambda_i)) \sqcup (\sqcup_{i=1}^k C_i)) / \sim.$$

Here $C_i = S(\Delta_i) \times [0, 1]$ (if $\Delta_i = \emptyset$, then $S(\Delta_i)$ is one point), and we glue one end of C_i to $S(\Delta_i) \subset S(\Gamma_2)$, and another end to $S(\Delta_i) \subset S(\Lambda_i)$. There is a homotopy equivalence $\bar{S}(\Gamma) \rightarrow S(\Gamma)$ by collapsing the $[0, 1]$ factor of each C_i , which lifts to a map between their universal covers $r : \tilde{X}(\Gamma) \rightarrow X(\Gamma)$. If an edge of $\tilde{X}(\Gamma)$ is not degenerate under r , then it inherits a label from its image, otherwise we leave it unlabelled. $\bar{S}(\Gamma)$ is non-positively curved ([15, Proposition II.11.13]), hence $\tilde{X}(\Gamma)$ is a $CAT(0)$ cube complex. Since the action of H on $X(\Gamma)$ preserves Λ_i -components and Γ_2 -components, there is an action $H \curvearrowright \tilde{X}(\Gamma)$ by automorphisms such that r is H -equivariant.

Let \mathcal{H} be the collection of hyperplanes in $\bar{X}(\Gamma)$ which comes from hyperplanes in C_i dual to the $[0, 1]$ factor. Note that different elements in \mathcal{H} do not intersect. Let T be the dual tree to the wallspace $(\bar{X}(\Gamma), \mathcal{H})$, i.e. each vertex of T corresponds to a component of $\bar{X}(\Gamma) \setminus \mathcal{H}$, and two vertices are adjacent if the corresponding two components are adjacent along an element of \mathcal{H} . Then there is an induced action $H \curvearrowright T$. Moreover, we have an H -equivariant cubical map $q : \bar{X}(\Gamma) \rightarrow T$ by collapsing edges which are not dual to hyperplanes in \mathcal{H} . Pick point $s \in T$. If s is a vertex, then $q^{-1}(s)$ is either a Λ_i -component or a Γ_2 -component. If s is not a vertex, then $q^{-1}(s)$ is isometric to a Euclidean space.

Up to passing to a subgroup of index 2, we assume H acts on T without inversion. Let $\mathcal{G} = T/H$. Then the natural map $\bar{X}(\Gamma)/H = K \rightarrow \mathcal{G}$ induces a graph of spaces decomposition of the orbifold K . Moreover, any cover of K has an induced graph of spaces decomposition. Since the action $H \curvearrowright \bar{X}(\Gamma)$ maps Λ_i -components to Λ_i -components for each i , and maps v -edges to v -edges for each vertex $v \in \Gamma_2$, it makes sense to talk about Λ_i -components and Γ_2 -components in K .

Let $K_v \subset K$ be a vertex space. A finite sheet cover K'_v of K_v is *good* if the following holds. If K_v is a Γ_2 -component, then K'_v is good if it is torsion free and special. Corollary 7.11 implies such cover exists. Suppose K_v is a Λ_i -component. Then $K_v = X_v/G_v$, where X_v is a Λ_i -component in $\bar{X}(\Gamma)$ and $G_v \leq H$ is the stabilizer of X_v . X_v has a product decomposition $X_v = X(\Delta_i) \times X(\Gamma_{1,\Delta_i})$. Note that G_v maps w -edges to w -edges for each vertex $w \in \Delta_i$, however, this may not be true if $w \in \Gamma_{1,\Delta_i}$. K'_v is good if it corresponds to a subgroup $G'_v \leq G_v$ such that (1) $G'_v = \mathbb{Z}^n \times L'_v$ where \mathbb{Z}^n acts trivially on the $X(\Gamma_{1,\Delta_i})$ and L'_v acts trivially on the $X(\Delta_i)$ factor, here n is the number of vertices in Δ_i ; (2) there exists a L'_v -invariant admissible labelling of $X(\Gamma_{1,\Delta_i})$. Since $X(\Delta_i) \cong \mathbb{E}^n$, we can apply Lemma 4.10 n times to deduce the existence of cover satisfying (1). Since Γ_{1,Δ_i} is a disjoint union of cliques, Corollary 8.8 implies that there exists a good cover of K_v .

Using Lemma 7.4 and the argument in Section 7.2.1, we can construct a finite cover $K' \rightarrow K$ such that each vertex space of K' is good. Then it is possible to put an admissible labelling on each Λ_i -component of $\bar{X}(\Gamma)$ in a way which is $\pi_1(K')$ -invariant. By applying the H -equivariant retraction map $r : \bar{X}(\Gamma) \rightarrow X(\Gamma)$, we obtain a $\pi_1(K')$ -invariant admissible labelling of $X(\Gamma)$. Up to conjugation, we can assume $\pi_1(K')$ preserves the reference labelling (Corollary 8.5). Then the theorem follows from Lemma 8.6. \square

9. INDUCED 4-CYCLE AND FAILURE OF COMMENSURABILITY

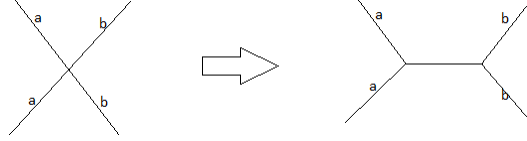
9.1. Label-preserving action on product of two trees. Pick two trees T and T' , and let H be a torsion free group acting geometrically on $T \times T'$ by automorphisms. H is *reducible* if it has a finite index subgroup which is a product of free groups, or equivalently, $(T \times T')/H$ has a finite sheet cover which is isomorphic to a product of two graphs. Otherwise H is *irreducible*.

Up to passing to a subgroup of index 2, we assume H does not flip the two tree factors. Then there are factors actions $H \curvearrowright T$ and $H \curvearrowright T'$.

Theorem 9.1. ([18]) *Let $h_1 : G \rightarrow \text{Aut}(T)$ and $h_2 : G \rightarrow \text{Aut}(T')$ be the homomorphisms induce by the factor actions. Then the following are equivalent.*

- (1) *The image of h_1 is a discrete subgroup of $\text{Aut}(T)$.*
- (2) *The image of h_2 is a discrete subgroup of $\text{Aut}(T')$.*
- (3) *G is reducible.*

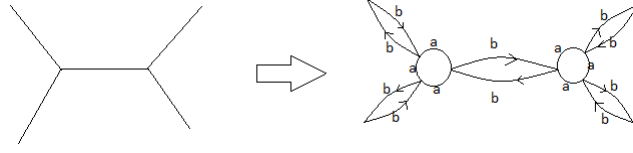
Let T_n be the n -valence tree. When Γ is a 4-gon, $X(\Gamma) \cong T_4 \times T_4$. Thus we can label edges of $T_4 \times T_4$ by vertices of Γ . It is known that there is an irreducible group acting on $T_4 \times T_4$ ([45]), however, such example is not label-preserving. In this section, we will construct an irreducible group acting on $T_4 \times T_4$ by label-preserving automorphisms. Given a label-preserving action of H on $T_4 \times T_4$, we can pass to an action of H on $T_3 \times T_3$ by modifying each T_4 as follows.



The other direction is given as follows.

Lemma 9.2. *Suppose there exists an irreducible group H acting geometrically on $T_3 \times T_3$. Then there exists an irreducible group H' acting geometrically on $T_4 \times T_4$ in a label-preserving way.*

Proof. We modify the first factor T_3 to obtain a 4-valence graph \mathcal{G} as follows.



More precisely, we replace each vertex by a 3-cycle with its edges labelled by a , and double each edge to obtain two edges labelled by b . We also orient each b -edge of \mathcal{G} such that the orientations on two consecutive b -edges are consistent. Then each automorphism of T_3 induces a unique label-preserving automorphism of \mathcal{G} which respects the orientation of b -edges. Thus the factor action $H \curvearrowright T_3$ induces a label-preserving action $H \curvearrowright \mathcal{G}$. In particular, we obtain an action $H \curvearrowright \mathcal{G} \times T_3$. This action is geometric, since there is an H -equivariant map $\mathcal{G} \times T_3 \rightarrow T_3 \times T_3$ induced by the natural map $\mathcal{G} \rightarrow T_3$. Let $G = \pi_1((\mathcal{G} \times T_3)/H)$. Then G acts geometrically on $T_4 \times T_3$, and there is an exact sequence $1 \rightarrow \pi_1(\mathcal{G} \times T_3) \rightarrow G \rightarrow H \rightarrow 1$. By comparing the action of H and G on the second tree factor, we deduce the irreducibility of G from the irreducibility of H and Theorem 9.1. We can modify the second tree factor in a similar way. \square

Now we construct an irreducible group acting on $T_3 \times T_3$. First we consider the following modification of the main example in [60]. Let X be the graph in the top of Figure 9.2.1 below. The top left and top right picture indicate two different ways of labelling and orientating edges of X . Let Y be the graph in the bottom. Then there are two different covering maps $f_1 : X \rightarrow Y$ and $f_2 : X \rightarrow Y$ induced by the edge labelling and orientation in the top left and top right respectively.

Let $Z' = (X \times [0, 1] \sqcup Y \times [0, 1]) \sim$, where for $i = 0, 1$, we identify $X \times \{i\}$ with $Y \times \{i\}$ via the covering map f_{i+1} (i.e. x and $f_{i+1}(x)$ are identified). One readily verify that with the natural cube complex structure on Z' , the universal cover of Z' is isomorphic to $T_3 \times T_3$. Now we collapse the $[0, 1]$ factor in $Y \times [0, 1]$, which gives a homotopy equivalence $Z' \rightarrow Z$. A minor adjustment of the argument in [60, Chapter II.3.2] implies $\pi_1(Z)$ is irreducible, hence $\pi_1(Z')$ is also irreducible. For

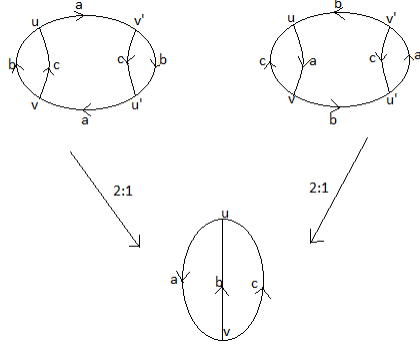


FIGURE 9.2.1.

the convenience of the reader, we give a detailed proof of the irreducibility of $\pi_1(Z)$ in the appendix.

Theorem 9.3. *There exists a torsion free irreducible group acting geometrically on $T_3 \times T_3$. Hence there exists a torsion free irreducible group acting geometrically on $T_4 \times T_4$ by label-preserving automorphisms.*

9.2. The general case. Let G be a group. Recall that the *profinite topology* on G is the topology generated by cosets of its finite index subgroup. A subset $K \subset G$ is *separable* if it is closed in the profinite topology. Equivalently, K is separable if for any $g \notin K$, there exists a finite index normal subgroup $N \triangleleft G$ such that $\phi(g) \notin \phi(K)$ for $\phi : G \rightarrow G/N$.

G is *residually finite* if the identity element is closed under profinite topology. Being residually finite is a commensurability invariant. If G is residually finite, then the solution to any equation on G is separable, since the group multiplication is continuous with respect to the profinite topology. In particular, the centralizer of every element in G is separable.

Lemma 9.4. *Let Γ be a 4-gon and let H be an irreducible group acting geometrically on $X(\Gamma)$ by label-preserving automorphisms. Then H is not residually finite.*

Proof. We argue by contradiction and assume H is residually finite. For each vertex $v \in \mathcal{P}(\Gamma)$, let P_v be the v -parallel set defined in Definition 2.5 and let $H_v \leq H$ be the stabilizer of P_v . Since H is label-preserving, H_v acts on P_v cocompactly. Let $P_v = \ell_v \times \ell_v^\perp$ be the product decomposition of P_v , where ℓ_v is a standard geodesic line with $\Delta(\ell_v) = v$.

Note that there exists a finite index subgroup $H'_v \leq H_v$ such that $H'_v = \mathbb{Z} \oplus L_v$ with \mathbb{Z} acting trivially on the ℓ_v^\perp factor and L_v acting trivially on the \mathbb{Z} factor. Let g be a generator of the \mathbb{Z} factor of H'_v . Then the centralizer $C_H(g)$ of g in H is of finite index in H_v . Since H is residually finite, $C_H(g)$ is separable in H , hence there exists a finite index subgroup $H_1 \leq H$ such that $H_1 \cap H_v = C_H(g)$. Note that the factor action $C_H(g) \curvearrowright \ell_v$ does not flip the two ends of ℓ_v .

Since there are only finitely many H -orbits of parallel sets of standard lines, we can repeat the above argument finitely many times to obtain a finite index subgroup $H' \leq H$ such that for each vertex $v \in \mathcal{P}(\Gamma)$, the factor action $H'_v \curvearrowright \ell_v$ does not flip the two ends of ℓ_v . In this case, we can orient each standard line in $X(\Gamma)$ in an H' -invariant way such that they are compatible with parallelism. Thus there

exists $k \in \text{Aut}(X(\Gamma))$ such that $kH'k^{-1}$ preserves both the reference labelling and the reference orientation, which implies H' is reducible. \square

Theorem 9.5. *Let Γ be any finite simplicial graph which contains an induced 4-cycle. Then there exists a group H acting geometrically on $X(\Gamma)$ by label-preserving automorphisms such that it is not residually finite. In particular, H is not commensurable to $G(\Gamma)$.*

Proof. Let $\Gamma' \subset \Gamma$ be an induced 4-cycle and pick a Γ' -component $Y \subset X(\Gamma)$. The covering map $X(\Gamma) \rightarrow S(\Gamma)$ induces a local isometry $Y \rightarrow S(\Gamma)$. Let Z be the canonical completion of $Y \rightarrow S(\Gamma)$. Then the universal cover of Z is isomorphic to $X(\Gamma)$. Let H be a torsion free irreducible group acting on Y in a label-preserving way (Theorem 9.3). The $H \curvearrowright Y$ extends to a label-preserving free action $H \curvearrowright Z$ such that the inclusion $Y \rightarrow Z$ is H -equivariant. Let $G = \pi_1(Z/H)$. Then G acts geometrically on $X(\Gamma)$ by label-preserving automorphisms. Moreover, the inclusion $Y \rightarrow X(\Gamma)$ induces an embedding $H \rightarrow G$. Since H is not residually finite, so is G . However, each RAAG is residually finite [36], thus H and $G(\Gamma)$ are not commensurable. \square

Corollary 9.6. *Let Γ be any finite simplicial graph which contains an induced 4-cycle. Then there exists a group H quasi-isometric to $G(\Gamma)$ such that it is not commensurable to $G(\Gamma)$.*

APPENDIX A. IRREDUCIBILITY OF Z

We show the fundamental group of Z in Section 9.1 is irreducible. We will follow [60, Chapter II.3.2] very closely.

Let X, Y, f_1, f_2 be as in Section 9.1. Note that there is a cubical map $r : X \times [0, 1] \rightarrow Z$. An edge in $X \times [0, 1]$ is *vertical* if it is parallel to the X factor, otherwise it is *horizontal*. An edge in Z is called *vertical* or *horizontal* if it is the r -image of a vertical or horizontal edge in $X \times [0, 1]$. The *vertical* or *horizontal 1-skeleton* of Z , denoted by $Z_v^{(1)}$ or $Z_h^{(1)}$, is the union of all vertical edges or horizontal edges in Z . We identify the vertical 1-skeleton $Z_v^{(1)}$ with Y , hence edges in $Z_v^{(1)}$ inherit orientation and labelling from Y . The two vertices of Z are in $Z_v^{(1)}$, and we also denote them by u and v .

Recall that X has 4 vertices $\{u, v, u', v'\}$ (see Figure 9.2.1), which gives 4 horizontal edges in $X \times [0, 1]$. The r -image of them are 4 horizontal loops in Z , which we denote by $e_u, e_v, e_{u'}, e_{v'}$ respectively. The horizontal 1-skeleton of Z has two connected components, one of form $e_u \cup e_{u'}$, and the other of form $e_v \cup e_{v'}$.

Then Z has a graph of spaces decomposition, where the underlying graph is a circle with one vertex, the edge space is X , the vertex space is Y , and the two boundary morphisms are f_1 and f_2 respectively.

Note that if Z has a finite cover which is a product of two graphs, then every pair of vertical edge loop ℓ_1 (i.e. an edge loop made of vertical edges) and horizontal edge loop ℓ_2 should virtually commute in $\pi_1(Z)$, i.e. ℓ_1^n and ℓ_2^m commute for some non-zero integer n and m . So it suffices to find a vertical loop and a horizontal loop which do not virtually commute.

We assume ℓ_1 and ℓ_2 are locally geodesic. Pick a lift \tilde{v} of v in the universal cover \tilde{Z} of Z . For $i = 1, 2$, we lift the path ℓ_i^∞ (which is the concatenation of countably infinitely many ℓ_i 's) to a geodesic ray $\tilde{\ell}_i$ in \tilde{Z} emanating from \tilde{v} . Then $\tilde{\ell}_1$ and $\tilde{\ell}_2$

span a quarter-plane. We identify this quarter plane with $[0, \infty) \times [0, \infty)$ such that $\ell_1 = \{0\} \times [0, \infty)$. For integers $n, m \geq 0$, we define $\varphi_n(\ell_1^m) = p(\{n\} \times [0, m])$ and $\varphi_n(\ell_1^\infty) = p(\{n\} \times [0, \infty))$ where $p: \tilde{Z} \rightarrow Z$.

Let $\omega \subset Y$ be an edge path based at v . There are two lifts of ω with respect to f_1 , based at v and v' respectively. We map them back in Y via f_2 and denote the resulting edge paths by $\phi_1(\omega)$ and $\phi_2(\omega)$ respectively. If $\omega = \ell_1$ is a locally geodesic loop and $\ell_2 = e_v$, then for any $n, m \geq 0$,

$$(A.1) \quad (\phi_1)^n(\omega^m) = \varphi_n(\omega^m).$$

Each ω gives rise to a word on $\{a^\pm, b^\pm, c^\pm\}$ (however, an arbitrary word may not give rise to an edge path based at v). Let $\sharp_a(\omega)$ be the number of a 's in ω (counted with sign). We define $\sharp_b(\omega)$ and $\sharp_c(\omega)$ in a similar way.

Let ι be the automorphism of Y that flips the a -edge and c -edge, and fixes the b -edge. For $i = 1, 2$, let G_i be the subgroup of $\pi_1(Y, v)$ induced by $f_i: X \rightarrow Y$. We record the following observations.

Lemma A.2. *Let σ be an edge loop in Y based at v .*

- (1) $\sigma \in G_1 \Leftrightarrow \sharp_b(\sigma) + \sharp_c(\sigma) \equiv_2 0$
- (2) $\phi_2 = \iota \circ \phi_1$
- (3) $\sigma \notin G_1 \Rightarrow \phi_1(\sigma^2) = \phi_1(\sigma) \cdot \phi_2(\sigma)$

Lemma A.3. *Let σ be an edge loop based at v . If $\sigma \notin G_1$, then $\phi_1(\sigma^2) \notin G_1$.*

Proof. By Lemma A.2 (1), it suffices to prove $\sharp_b(\phi_1(\sigma^2)) + \sharp_c(\phi_1(\sigma^2)) \equiv_2 1$.

$$\begin{aligned} \sharp_b(\phi_1(\sigma^2)) + \sharp_c(\phi_1(\sigma^2)) &\equiv_2 \sharp_a(\phi_1(\sigma^2)) = \\ \sharp_a(\phi_1(\sigma) \cdot \phi_2(\sigma)) &= \sharp_a(\phi_1(\sigma)) + \sharp_a(\iota \circ \phi_1(\sigma)) = \\ \sharp_a(\phi_1(\sigma)) - \sharp_c(\phi_1(\sigma)) &\equiv_2 \sharp_a(\phi_1(\sigma)) + \sharp_c(\phi_1(\sigma)) \equiv_2 \\ \sharp_b(\sigma) + \sharp_c(\sigma) &\equiv_2 1. \end{aligned}$$

The first equality holds since there are even number of edges in $\phi_1(\sigma^2)$. The second equality follows from Lemma A.2 (3). The third equality follows from Lemma A.2 (2). The fourth equality follows from the definition of ι . Note that for any edge $e \subset \sigma$, the label of e belongs to $\{b, c, b^{-1}, c^{-1}\}$ if and only if the label of $\phi_1(e)$ in $\phi_1(\sigma)$ belongs to $\{a, c, a^{-1}, c^{-1}\}$. Thus the sixth equality holds. The last equality follows from Lemma A.2 (1). \square

In the rest of this section, we always pick $\ell_2 = e_v$ in the definition of φ_n , and let σ be the vertical loop based at v of form ba (see Figure 9.2.1). We claim $\varphi_n(\sigma^{2^n}) \notin G_1$ for any integer $n \geq 1$. By (A.1), it suffices to prove $\phi_1^n((\sigma)^{2^n}) \notin G_1$. Since $\sigma \notin G_1$, then case $n = 1$ follows from Lemma A.3. Note that when $n > 1$, $\phi_1^{n-1}(\sigma^{2^n}) = (\phi_1^{n-1}(\sigma^{2^{n-1}}))^2$. To see this, we start with $\phi_1(\sigma^{2^n}) = (\phi_1(\sigma^2))^{2^{n-1}}$ (since $\sigma^2 \in G_1$) and iterate. Thus the claim follows by iterating Lemma A.3. The claim is also true if we replace σ by σ^k with k odd, since $\sigma^k \notin G_1$ in this case.

It suffices to show there do not exist integers $n, m \neq 0$ such that σ^m and e_v^n commute in $\pi_1(Z, v)$. Suppose σ^m and e_v^n commutes. We can assume $m, n > 0$. Suppose $m = k \cdot 2^l$ with k odd. Then $\varphi_{2nl}((\sigma^k)^{2^{2nl}}) \notin G_1$. Note that $\varphi_{2nl}((\sigma^k)^{2^{2nl}}) = \varphi_{2nl}(\sigma^{2^{2nl-l} \cdot m})$. Since σ^m and e_v^n commute, $\varphi_{2nl}(\sigma^{2^{2nl-l} \cdot m}) = \sigma^{2^{2nl-l} \cdot m} \in G_1$ (note that $2^{2nl-l} \cdot m$ is even), which is a contradiction.

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THE DEPARTMENT OF MATHEMATICS AND STATISTICS, MCGILL UNIVERSITY, BURNSIDE HALL,
805 SHERBROOKE W., MONTREAL, QC, H3A 0B9, CANADA
E-mail address: jingyin.huang@mcgill.ca